Regret-free truth-telling voting rules*

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Abstract

We study the ability of different classes of voting rules to induce agents to report their preferences truthfully, if agents want to avoid regret. First, we show that regret-free truth-telling is equivalent to strategy-proofness among tops-only rules. Then, we focus on three important families of (non-tops-only) voting methods: maxmin, scoring, and Condorcet consistent ones. We prove positive and negative results for both neutral and anonymous versions of maxmin and scoring rules. In several instances we provide necessary and sufficient conditions. We also show that Condorcet consistent rules that satisfy a mild monotonicity requirement are not regret-free truth-telling. Successive elimination rules fail to be regret-free truth-telling despite not satisfying the monotonicity condition. Lastly, we provide two characterizations for the case of three alternatives and two agents.

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1 Introduction

Voting rules are procedures that allow a group of agents to select an alternative, among many, according to their preferences. Naturally, their vulnerability to manipulation is a primary concern. Thus, it is desirable that voting rules are strategy-proof, meaning that it is always in the best interest of agents to report their preferences truthfully regardless of the behavior of others.

Unfortunately, outside of a dictatorship, there is no strategy-proof voting rule when more than two alternatives, and all possible preferences over alternatives, are considered (Gibbard, 1973; Satterthwaite, 1975). Despite this negative result, lack of strategy-proofness does not mean that a voting rule is easy to manipulate nor that agents have the information to safely do so. The required manipulation may have to be specifically tailored to the reported preferences of all other participants. Given that agents typically do not have access to such information in practice, we approach the incentives to manipulate from a different perspective; one that allows agents to evaluate the potential outcomes of a manipulation according to the partial information they hold and the inferences they can make based on what they observe.

We examine the incentives to report preferences truthfully from the lenses of regret avoidance, in the sense of Fernandez (2020). Regret is intimately connected to (i) choice, and (ii) counterfactuals; see Zeelenberg and Van Dijk (2007). We assume that an agent knows their own preference over alternatives and the voting rule used to select an outcome. Upon observing the outcome, the agent infers which were the possible preferences profiles reported by the other agents. An agent suffers regret if the chosen report is dominated ex-post. A voting rule is regret-free truth-telling if it guarantees that no agent regrets reporting their preferences truthfully.

First, we show that if the voting rule only considers the most preferred alternative reported by each agent (i.e. it is tops-only), then regret-free truth-telling is equivalent to strategy-proofness. This equivalence implies that: (i) for problems with only two alternatives, extended majority voting rules are the only regret-free truth-telling rules; and, (ii) there are no non-dictatorial tops-only regret-free truth-telling voting rules, on the universal domain. Thus, we examine non-tops-only rules when there are more than two alternatives. We study whether three of the most important families of non-tops-only voting methods satisfy regret-free truth-telling: (i) maxmin methods, (ii) scoring methods, and (iii) Condorcet consistent methods. Maxmin methods select those alternatives that "make the least happy agent(s) as happy as possible" (Rawls, 1971). Scoring methods assign points to each alternative according to the rank it has in agents' preferences and selects one of the alternatives with highest score. Condorcet consistent methods select the pairwise majority winner (Condorcet winner) whenever one exists. To resolve potential multiplicity in the scoring and maxmin methods we consider two classical tie-breakings. One is defined by picking the preference of a fixed agent (neutral). The other is defined by a fixed order of the alternatives (anonymous).

Given n agents and m alternatives, we show that all neutral maxmin rules are regret-free truth-telling. Anonymous maxmin rules are regret-free truth-telling if and only if $n \geq m-1$ or n divides $m-1$. We also obtain general positive results for the negative plurality rule, a special scoring rule in which all the rank positions get one point except the last one which gets zero. The results are analogous to those of maxmin rules: all neutral negative plurality rules are regret-free truth-telling, whereas an anonymous negative plurality rule is regret-free truth-telling if and only if $n \geq m - 1$.

For general scoring rules, we provide necessary and sufficient conditions for some classes of these rules to be regret-free truth-telling. The conditions depend intimately on the highest position/rank where the score is not maximal. Among others, these include k-approval rules. Notably, Borda, plurality and Dowdall rules, as well as all efficient and anonymous rules fail to be regret-free truth-telling.

We find that Condorcet consistent rules are incompatible with regret-free truth-telling under a mild monotonicity condition. The monotonicity condition says that if an alternative is ranked below the outcome of the rule for an agent and he changes his preferences modifying only the ordering of alternatives ranked above the outcome, then such alternative continues not to be chosen. In particular, we get that the six famous Condorcet consistent rules associated with the names of Simpson, Copeland, Young, Dodgson, Fishburn and Black (in both anonymous and neutral versions) are monotone, and therefore not regret-free truth-telling.[1] We also show that a family of non-monotone Condorcet consistent rules, the successive elimination rules, are not regret-free truth-telling either.

Finally, for the case with two agents and three alternatives, we present two characterization results. The first one says that a rule is regret-free truth-telling, efficient, and anonymous if and only if it is either a successive elimination or an anonymous maxmin rule in which the tie-breaking device is an antisymmetric and complete (not necessarily transitive) binary relation. The second one says that a rule is regret-free truth-telling and neutral if and only if it is a dictatorship or a maxmin rule with a specific type of tie-breaking that preserves neutrality.

1.1 Related literature

Two main approaches have been taken to circumvent Gibbard-Satterthwaite's impossibility theorem. The first approach restricts the domain of preferences that agents can have over alternatives (see Barberà, 2011, and references therein). This paper contributes to the literature following the second approach, that is, to consider different notions of strategic behavior.[2]

Among the papers considering weakenings of strategy-proofness, a recent literature has emerged that incorporates the (possibly) partial information held by agents. Notably, Reijngoud and Endriss (2012) and Endriss et al. (2016), Gori (2021), and Troyan and Morrill (2020) and Aziz and Lam (2021), which we discuss in turn.

Reijngoud and Endriss (2012) and Endriss et al. (2016) study when an agent has an incentive to manipulate different voting rules subject to different information functions. Importantly, there, the concept of winner information function leads to a property equivalent to regret-free truth-telling.

[1] For the anonymous Simpson and Copeland rules, these results have been previously obtained by Endriss et al. (2016).

[2] Early examples of this approach include Farquharson (1969)'s sophisticated voting, Moulin (1979)'s dominance-solvable voting schemes, as well as Barberà and Dutta (1982) protective strategies and Moulin (1981) prudent strategies.

Recently, Gori (2021) studies in detail a special case of an information function, where the information about the preferences of the other individuals in the society is limited to the knowledge, for every pair of alternatives, of the number of people preferring the first alternative to the second one. This notion is called WMG-strategy-proofness by Gori (2021). In that paper, there are a positive result showing a class of Pareto optimal, WMG-strategy-proof and non-dictatorial voting functions; and a negative result proving that, when at least three alternatives are considered, no Pareto optimal and anonymous voting function is WMG-strategy-proof.

Troyan and Morrill (2020) introduce the concept of obvious manipulation in the context of market design, while Aziz and Lam (2021) apply it in the context of voting. Troyan and Morrill (2020) assume that an agent knows the possible outcomes of the mechanism conditional on his own declaration of preferences, and define a deviation from the truth to be an obvious manipulation if either the best possible outcome under the deviation is strictly better than the best possible outcome under truth-telling, or the worst possible outcome under the deviation is strictly better than the worst possible outcome under truth-telling. A mechanism that does not allow any obvious manipulation is called not-obviously-manipulable. Aziz and Lam (2021) present a general sufficient condition for a voting rule to be not-obviously-manipulable. They show that Condorcet consistent as well as some other strict scoring rules are not-obviously-manipulable. Furthermore, for the class of k-approval voting rules, they give necessary and sufficient conditions for obvious manipulability.

The rest of the paper is organized as follows. In Section 2, we introduce the model and the property of regret-free truth-telling. We show the equivalence of regret-free truth-telling and strategy-proofness for tops-only rules in Section 3, where we also characterize extended majority voting rules as the only regret-free truth-telling rules when there are only two alternatives to choose from. In Section 4.1, we provide necessary and sufficient conditions for maxmin rules to be regret-free truth-telling. Section 4.2 we provide positive and negative results regarding scoring rules. In Section 4.3, we present negative results for Condorcet consistent rules. The special case with two agents and three alternatives is analyzed in Section 5, where two characterizations are presented.

2 Preliminaries

2.1 Model

A set of *agents* $N = \{1, \ldots, n\}$, with $n \geq 2$, has to choose an alternative from a finite and given set X, with $|X| = m \geq 2$. Each agent $i \in N$ has a strict *preference* P_i over X. We denote by R_i the weak preference over X associated to P_i; i.e., for all $x, y \in X$, xR_iy if and only if either $x = y$ or xP_iy. Let $\mathcal{P}$ be the set of all strict preferences over X. A *(preference) profile* is a n-tuple $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$, an ordered list of n preferences, one for each agent. Given a profile P and an agent i, P_{-i} denotes the subprofile obtained by deleting P_i from P. For each $P_i \in \mathcal{P}$, denote by $t_k(P_i)$ to the alternative in the *k-th position* (from bottom to top). Many times we write $t(P_i)$ instead of $t_m(P_i)$, and refer to it as the *top* of P_i.

We often also write P_i as an ordered list

$$P_i : t_m(P_i), t_{m-1}(P_i), \dots, t_1(P_i).$$

A *(voting) rule* is a function $f : \mathcal{P}^n \to X$ that selects for each preference profile $P \in \mathcal{P}^n$ an alternative $f(P) \in X$. We assume throughout that a voting rule is an onto function. Next, we define several classical properties that a rule may satisfy and that we use in the sequel. A rule f is:

- *strategy-proof* if agents can never induce a strictly preferred outcome by misrepresenting their preferences; namely, for each $P \in \mathcal{P}^n$, each $i \in N$ and each $P'_i \in \mathcal{P}$,
$$f(P) R_i f(P'_i, P_{-i});$$
- *efficient* if, for each $P \in \mathcal{P}^n$, there is no $y \in X$ such that $y P_i f(P)$ for each $i \in N$;
- *tops-only* if $P, P' \in \mathcal{P}^n$ such that $t(P_i) = t(P'_i)$ for each $i \in N$ imply $f(P) = f(P')$;
- *dictatorial* if there exists $i \in N$ such that for each $P \in \mathcal{P}^n$, $f(P) = t(P_i)$. In a dictatorial rule, in each profile of preferences, the same agent selects his most preferred outcome;
- *unanimous* if $t(P_i) = x$ for each $i \in N$ imply $f(P) = x$. Unanimity is a natural and weak form of efficiency: if all agents consider an alternative as being the most-preferred one, the rule should select it;
- *anonymous* if for each $P \in \mathcal{P}^n$ and each bijection $\pi : N \to N$, $f(P) = f(P^\pi)$ where for each $i \in N$, $P^\pi_i = P_{\pi(i)}$. Anonymity requires that the rule treats all agents equally because the social outcome is selected without paying attention to the identities of the agents;
- *neutral* if for each $P \in \mathcal{P}^n$ and each bijection $\pi : X \to X$, $\pi(f(P)) = f(\pi P)$ where $\pi P_i : \pi(t(P_i)), \pi(t_{m-1}(P_i)), ..., \pi(t_1(P_i))$.

In general, the axioms of anonymity and neutrality are incompatible for voting rules. A classical way to address such incompatibility is to consider rules defined in two stages as follows:[3]

(i) First, consider a *voting correspondence* $\mathcal{Y} : \mathcal{P}^n \to 2^X \setminus \{\emptyset\}$ that for each preference profile $P \in \mathcal{P}^n$ chooses a (non-empty) subset $\mathcal{Y}(P) \subseteq X$, and assume that $\mathcal{Y}$ satisfies both anonymity and neutrality.

(ii) Second, given $P \in \mathcal{P}^n$, consider a strict order on X and choose the maximal element according to that order in $\mathcal{Y}(P)$. There are two classical selections of such an order, one to preserve anonymity and the other to preserve neutrality:

[3] "In practice, we will be happy with (voting) correspondences that respect the three principles (efficiency, anonymity and neutrality). If a deterministic election is called for, we will use either a non-anonymous tie-breaking rule or a non-neutral one" (see Moulin, 1991, p.234).

(a) The strict order $\succ$ is independent of P and is part of the rule's definition. In this case anonymity is preserved and the rule is defined by[4]

$$f(P) = \max_{\succ} \mathcal{Y}(P). \tag{1}$$

(b) There exists an agent $i \in N$ such that, for each $P \in \mathcal{P}$, the strict order we consider is P_i. In this case, neutrality is preserved and the rule is defined by

$$f(P) = \max_{P_i} \mathcal{Y}(P). \tag{2}$$

From Section 4 onward, we study rules that can be defined by this two-stage procedure. This way to define a rule is flexible enough to encompass many well-known and long-studied families of rules.

2.2 Regret-free truth-telling

A regret-free truth-telling rule provides incentives to report preferences truthfully if agents want to avoid regret. When observing an outcome, an agent regrets his report if it is dominated ex-post. That is, if there is an alternative report that would have guaranteed him the same or a better outcome for any preference report of others, consistent with his observation. We can then define a rule to be regret-free truth-telling as one in which no agent ever regrets reporting their preferences truthfully. Equivalently, given an observed outcome after reporting his preferences truthfully, if the agent could have done better through an alternative report for some configuration of reports of others consistent with his observation, then, that same alternative report could have done worse for another configuration of reports consistent with his observation. Formally,

Definition 1. *The rule $f : \mathcal{P}^n \to X$ is **regret-free truth-telling** if for each $i \in N$, each $P \in \mathcal{P}^n$, and each $P_i' \in \mathcal{P}$ such that $f(P_i', P_{-i}) P_i f(P)$, there is $P^\star_{-i} \in \mathcal{P}^{n-1}$ such that*

$$f(P_i, P^\star_{-i}) = f(P) \text{ and } f(P_i, P^\star_{-i}) P_i f(P_i', P^\star_{-i}).$$

3 Tops-only rules and the case of two alternatives

It is clear that if a rule is strategy-proof, then it is also regret-free truth-telling. Our first result states that the converse is also true for tops-only rules.[5]

Proposition 1. *If a rule is regret-free truth-telling and tops-only, then it is strategy-proof.*

Proof. Let $f : \mathcal{P}^n \to X$ be a regret-free truth-telling and tops-only rule. Assume f is not strategy-proof. Then, there are $P \in \mathcal{P}^n$, $i \in N$, and $P_i' \in \mathcal{P}$ such that $f(P_i', P_{-i}) P_i f(P)$. Let

[4]Throughout the paper, given a strict order $>$ defined on a set A and a subset $B \subseteq A$, we denote by $\max_{>} B$ to the maximum element in set B according to order $>$.

[5]The result holds regardless of whether the voting rule is onto.

$\widetilde{P}_i \in \mathcal{P}$ be such that $t(\widetilde{P}_i) = t(P_i)$ and $t_1(\widetilde{P}_i) = f(P)$. Since f is tops-only, $f(\widetilde{P}_i, P_{-i}) = f(P)$. Therefore, since $t_1(\widetilde{P}_i) = f(P)$, it follows that

$$f(P'_i, P_{-i})\widetilde{P}_i f(\widetilde{P}_i, P_{-i}). \tag{3}$$

Let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $f(\widetilde{P}_i, P^\star_{-i}) = f(\widetilde{P}_i, P_{-i})$. Since $t_1(\widetilde{P}_i) = f(P) = f(\widetilde{P}_i, P_{-i})$, we have

$$f(P'_i, P^\star_{-i})\widetilde{R}_i f(\widetilde{P}_i, P^\star_{-i}). \tag{4}$$

By (3) and (4), f is not regret-free truth-telling. □

The result, in turn, leads to a complete characterization of the class of regret-free truth-telling rules for the case of two alternatives. In order to present it, we first need to define the family of extended majority voting rules on $\{x, y\}$.[6] Fix $w \in \{x, y\}$ and let 2^N denote the family of all subsets of N, referred to as *coalitions*. A family $\mathcal{C}_w \subseteq 2^N$ of coalitions is a *committee for w* if it satisfies the following monotonicity property: $S \in \mathcal{C}_w$ and $S \subsetneq T$ imply $T \in \mathcal{C}_w$. The elements in $\mathcal{C}_w$ are called *winning coalitions (for w)*.

Definition 2. *A rule $f : \mathcal{P}^n \to \{x, y\}$ is an **extended majority voting** rule if there is a committee $\mathcal{C}_x$ for x with the property that, for each $P \in \mathcal{P}^n$,*

$$f(P) = x \text{ if and only if } \{i \in N : t(P_i) = x\} \in \mathcal{C}_x.$$

The following corollary provides the characterization result.

Corollary 1. *Assume $m = 2$. Then,*

(i) A rule is regret-free truth-telling if and only if it is strategy-proof;

(ii) A rule is regret-free truth-telling if and only it is an extended majority voting rule.

Proof. (i) If f is strategy-proof it is clear that f is regret-free truth-telling. If f is regret-free truth-telling, since when $m = 2$ every rule is tops-only, f is strategy-proof by Proposition 1. (ii) It follows from (i) and Moulin (1980). □

Corollary 2. *Assume $m > 2$. A rule is regret-free truth-telling and tops-only if and only if it is a dictatorship.*

Proof. It follows from Proposition 1 and Gibbard-Satterthwaite's Theorem. □

4 Non-tops-only rules and more than two alternatives

From now on, we assume that $m > 2$.

[6]These rules are equivalent to the ones presented in Moulin (1980), where fixed ballots are used to describe them instead of committees.

4.1 Maxmin rules

Voting rules in class of maxmin methods select those alternatives that "make the least happy agents as happy as possible" (Rawls, 1971). Given $P \in \mathcal{P}^n$ and $x \in X$, the *minimal position* of x according to P is defined by

$$mp(x, P) = \min\{k : \text{there exists } i \in N \text{ such that } x = t_k(P_i)\}.$$

An alternative is a *maxmin winner* if there is no other alternative with higher minimal position. We denote the set of *maxmin winners* according to P as $\mathcal{M}(P)$. Namely,

$$\mathcal{M}(P) = \{x \in X : mp(x, P) \geq mp(y, P) \text{ for each } y \in X\}.$$

The idea of making the least happy agents as happy as possible is captured by rules that pick, for each preference profile, a maxmin winner for that profile. We study both anonymous and neutral versions of these rules. Formally,

Definition 3. *A rule $f : \mathcal{P}^n \to X$ is*

(i) ***A-maxmin*** *if there is a strict order $\succ$ on X such that, for each $P \in \mathcal{P}^n$,*

$$f(P) = \max_{\succ} \mathcal{M}(P).$$

(ii) ***N-maxmin*** *if there is an agent $i \in N$ such that, for each $P \in \mathcal{P}^n$,*

$$f(P) = \max_{P_i} \mathcal{M}(P).$$

The following theorem summarizes the positive results concerning regret-free truth-telling for these rules:

Theorem 1.

(i) An A-maxmin rule is regret-free truth-telling if and only if $n \geq m - 1$ or n divides $m - 1$.

(ii) Any N-maxmin rule is regret-free truth-telling.

Proof. See Appendix A.1. □

The idea behind the proof can be explained as follows. First, we analyze A-maxmin rules. To see that regret-free truth-telling implies $n \geq m - 1$ or n divides $m - 1$, consider an A-maxmin rule in which the tie-breaking has a and b as first and second alternatives, respectively, and alternative z as the last one.

Suppose that agent 1 has the preference $P_1 : b, \ldots, a, z, \ldots$, that he reports truthfully and that the outcome of the rule is a. Among the subprofiles consistent with a, there are some in which alternatives a and b are maxmin winners with respect to that profile, and the outcome is a due to the tie-breaking. For any such subprofile, agent 1 could have generated a better outcome by interchanging the order of alternatives a and b in his report. Namely, a would not longer be among the maxmin winners, whereas b would remain a maxmin winner and be selected by the rule due to the tie-breaking. The subprofiles where both a and b are maxmin winners, are not the only subprofiles consistent with observing the outcome of the rule being a. However, for any subprofile consistent with outcome a, agent's 1 misrepresentation by interchanging a and z results in maxmin winners that are either z or an alternative that is at least as good a according to agent 1's preferences.

The main claim is to prove that if neither $n \geq m-1$ nor n divides $m-1$, for any consistent subprofile there exists a maxmin winner different from z when agent 1 misrepresents his preferences by interchanging a and z. Therefore, as z is the last alternative in the tie-breaking and, as we said, the maxmin winners different from z under the misrepresentation and any consistent subprofile are always at least as good as a in the true preference, the agent regrets truth-telling.

Next, we argue that A-maxmin rules such that $n \geq m-1$ or n divides $m-1$ and N-maxmin rules are regret-free truth-telling. First, in both cases we prove that in any profitable misrepresentation the outcome of the rule under truth-telling, $f(P)$, has to be in a lower position than the one it has in the true preference of the agent. Therefore, in the misrepresentation there is an alternative x that is less preferred than $f(P)$ in the true preference that is lifted to a position greater or equal to the one $f(P)$ has in the true preference.

For each case, we construct a subprofile of the other agents such that: (i) it is consistent with the true preference of the agent and the outcome $f(P)$, and (ii) x is the only maxmin winner under the misrepresentation and the subprofile. Thus, the agent does not regret truth-telling.

For an A-maxmin rule such that $n \geq m-1$ the subprofile for the other agents is such that x and $f(P)$ are the first and second alternatives for each agent, respectively and for each alternative different from x and $f(P)$ there is an agent that has that alternative as his bottom alternative.

For an A-maxmin rule such that n divides $m-1$, the definition of the subprofile is more complicated but a similar argument to the one presented in the previous case can be performed.

Finally, for a N-maxmin rule the subprofile for the other agents is such that $f(P)$ and x are the first and second alternatives for each agent, respectively, while all the other alternatives keep their relative rankings. The technical details can be found in Appendix A.1.

4.2 Scoring rules

Next, we present the family of scoring rules. Given $P \in \mathcal{P}^n$ and $x \in X$, let $N(P,k,x) = \{i \in N : t_k(P_i) = x\}$ be the set of agents that have alternative x in the k-th position (from bottom to top) in their preferences, and let $n(P,k,x) = |N(P,k,x)|$. Let s_k be the score associated to the k-th position (from bottom to top) with $s_1 \leq s_2 \leq \ldots \leq s_m$ and $s_1 < s_m$. The *score of $x \in X$ according to P* is defined by

$$s(P,x) = \sum_{k=1}^{m} [s_k \cdot n(P,k,x)].$$

The set of *scoring winners* according to P is

$$\mathcal{S}(P) = \{x \in X : s(P, x) \geq s(P, y) \text{ for all } y \in X\}.$$

For future reference, given scores $s_1 \leq s_2 \leq \ldots \leq s_m$, let denote by $k^\star$ the highest position where the score is not maximal, i.e., $k^\star$ is such that $s_1 \leq s_2 \leq s_{k^\star} < s_{k^\star+1} = \ldots = s_m$.

Definition 4. *A rule $f : \mathcal{P}^n \to X$ is*

(i) ***A-scoring*** *associated to $s_1 \leq s_2 \leq \ldots \leq s_m$ if there is an order $\succ$ on X such that, for each $P \in \mathcal{P}^n$,*

$$f(P) = \max_{\succ} \mathcal{S}(P).$$

(ii) ***N-scoring*** *associated to $s_1 \leq s_2 \leq \ldots \leq s_m$ if there is an agent $i \in N$ such that, for each $P \in \mathcal{P}^n$,*

$$f(P) = \max_{P_i} \mathcal{S}(P).$$

Remark 1. *Some of the most well known scoring rules are:*

(i) the ***Borda*** *rule, in which $s_k = k$ for $k = 1, \ldots, m$;*

(ii) the ***Dowdall*** *rule, in which $s_k = \frac{1}{m-k+1}$ for $k = 1, \ldots, m$;*

(iii) the ***k-approval*** *rules, in which $0 = s_1 = s_2 = \ldots = s_{m-k}, s_{m-k+1} = \ldots = s_{m-1} = s_m = 1$ for some k such that $m - 1 \geq k \geq 1$, i.e., the top k scores are 1 and the rest are 0. In these rules, agents are asked to name their k best alternatives, and the alternative with most votes wins. Note that in this rule $k^\star = m - k$.*

Within these rules, two subclasses stand out:

(iii.a) the ***plurality*** *rule, where $k = 1$. Therefore $s_1 = s_2 = \ldots = s_{m-1} = 0$ and $s_m = 1$ (note that $k^\star = m - 1$);*

(iii.b) the ***negative plurality*** *rule, where $k = m - 1$. Therefore $s_1 = 0$ and $s_2 = \ldots = s_{m-1} = s_m = 1$ (note that $k^\star = 1$).*

Remark 2. *If an A-scoring rule is efficient, then $s_{m-1} < s_m$ (i.e., $k^\star = m - 1$).*

Observe that, by definition, $k^\star \in \{1, 2, \ldots, m - 1\}$. The next theorems consider the extreme cases in which $k^\star = 1$ and $k^\star = m - 1$, and allows us to present conclusive results about efficient A-scoring rules, the Borda rule, the Dowdall rule, as well as plurality and negative plurality rules.

Theorem 2.

(i) An A-scoring rule with $k^\star = 1$ (i.e., an A-negative plurality rule) is regret-free truth-telling if and only if $n \geq m - 1$.[7]

[7] For A-negative plurality rules, Theorem 6 in Reijngoud and Endriss (2012) presents a sufficient (though not necessary) condition ($n + 2 \geq 2m$) guaranteeing regret-free truth-telling. We present an independent proof that encompasses their result as well.

(ii) Any N-scoring rule with $k^\star = 1$ (i.e., any N-negative plurality rule) is regret-free truth-telling.

Proof. See Appendix A.2. □

The proof of Theorem 2 follows similar ideas to the ones that describe the proof of Theorem 1.

Theorem 3. *Assume $n > 2$. Then, no (anonymous or neutral) scoring rule with $k^\star = m - 1$ (i.e., $s_{m-1} < s_m$) is regret-free truth-telling.*

Proof. See Appendix A.3. □

We sketch the proof of Theorem 3 for the case of an odd number of voters. Consider a scoring rule in which the tie-breaking has a and b as first and second alternatives, respectively, in the anonymous case or agent 1 in the neutral case. Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	P_2	P_3	P_4	$\cdots$	P_{t+3}	P_{t+4}	$\cdots$	P_{2t+3}
a	c	b	a	$\cdots$	a	b	$\cdots$	b
c	b	a	b	$\cdots$	b	a	$\cdots$	a
b	a	c	c	$\cdots$	c	c	$\cdots$	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
				t agents			t agents	

Then, the alternatives a and b are scoring winners and the tie-breaking chooses a. Now, as $s_{m-1} < s_m$, agent 2 can manipulate the rule by interchanging b and c in his report, because the outcome of the rule then changes to b. Furthermore, for any profile where agent 2 declares his true preference and the outcome of the rule is a, if agent 2 interchanges b and c in his report then the outcome of the rule is at least as good as a. Hence, agent 2 regrets truth-telling.

Corollary 3. *Assume $n > 2$. Then, both anonymous and neutral versions of Borda, plurality, and Dowdall rules are not regret-free truth-telling. Moreover, no efficient A-scoring rule is regret-free truth-telling.*

From now on, we assume that $k^\star$ is such that $1 < k^\star < m - 1$. Next, we present some results for scoring rules by means of two complementary theorems, one of which can be considered as positive and the other one as negative. These theorems allow us to present conclusive results about approval rules and scoring rules in which $s_{k^\star-1} = s_{k^\star}$. Theorem 4 focuses on the case $k^\star n < m$, which encompasses the class of scoring rules where, in any preference profile, there is always an alternative that gets maximal score. This positive result gives a necessary and sufficient condition for an A-scoring rule to be regret-free truth-telling and also states that any N-scoring rule is regret-free truth-telling.

Theorem 4. *Assume that $n > 2$ and $k^\star n < m$. Then,*

(i) An A-scoring rule is regret-free truth-telling if and only if $k^\star n = m - 1$.

(ii) Any N-scoring rule is regret-free truth-telling.

Proof. See Appendix A.4. □

The main idea of the proof of Theorem 4 is the following. Assume that $n > 2$ and $k^\star n < m$. First, we analyze A-scoring rules. To see that regret-free truth-telling implies $k^\star n = m - 1$, consider a scoring rule in which the tie-breaking has a and b as first and second alternatives, respectively, and z as the last one. Assume $k^\star n < m - 1$. Then, for any profile of preferences there are at least two alternatives above position $k^\star$ in the preference of each agent. So, there are at least two score winners (with score equal to $n \cdot s_m$). Let $P \in \mathcal{P}^n$ be such that $P_1 : b, \ldots, a, z, \ldots$ where z is in the $k^\star$-th position and $P_j : a, b, \ldots$ for each $j \in N \setminus \{1\}$. Now, the argument to show that agent 1 regrets truth-telling is similar to the one used for Theorem 1 part (i).

Next, we argue that A-scoring rules such that $k^\star n = m - 1$ and N-scoring rules are regret-free truth-telling. In each case we prove that in any profitable misrepresentation of preferences the outcome of the rule under truth-telling, $f(P)$ (must above the $k^\star$-th position in the true preference) is moved to the $k^\star$-th position or below. Therefore, in the misrepresentation there is an alternative x, that is in the $k^\star$-th position or below in the true preference of the agent, that is lifted to a position greater than $k^\star$.

For an A-scoring rule such that $k^\star n = m - 1$, consider a profitable manipulation for an agent and a subprofile for the other agents such that x and $f(P)$ are the first and second alternatives for each agent, respectively, and for each alternative different from $f(P)$ and above the $k^\star$-th position in the true preference of the agent, there is an agent that has that alternative in the $k^\star$-th position or below. By the definition of the subprofile, $f(P)$ is the only alternative whose score is $n \cdot s_m$ and, therefore, it is the outcome under the true preference and that subprofile. Furthermore, as $f(P)$ is in position $k^\star$ or below in the misrepresentation, by the definition of the subprofile the outcome under the misrepresentation and the subprofile must necessarily be an alternative that is in the $k^\star$-th position or below in the true preference. Thus, the agent does not regret truth-telling.

For a N-scoring rule, first we prove that the tie-breaking agent does not manipulate. Then, consider a profitable manipulation for an agent and a subprofile for the other agents such that x and $f(P)$ are the first and second alternatives for each agent, respectively. By the definition of the subprofile and the tie-breaking agent, $f(P)$ is the outcome under the true preference and that subprofile. Furthermore, the outcome under the misrepresentation and the subprofile is x since it obtains score $n \cdot s_m$ and it is the first alternative for the tie-breaking agent.

Theorem 5 below gives a negative result for the case $k^\star n \geq m$ when $s_{k^\star-1} = s_{k^\star}$. When $s_{k^\star-1} \neq s_{k^\star}$, we believe that the existence of regret-free truth-telling rules in the class depends sensibly on the specific scores defining each rule.

Theorem 5. *Assume that $n > 2$ and $k^\star n \geq m$. Then, there is no regret-free truth-telling scoring rule (neither anonymous nor neutral) with $s_{k^\star-1} = s_{k^\star}$.*

Proof. See Appendix A.5. □

The main idea of the proof of Theorem 5 is the following. Assume that $n > 2$, $k^\star n \geq m$, and $s_{k^\star-1} = s_{k^\star}$. Consider a scoring rule in which the tie-breaking has a and b as first and second alternatives, respectively, in the case of an anonymous rule, or in which agent 1 breaks ties when the rule is neutral. As $k^\star n \geq m$, it follows that $k^\star(n-1) \geq m - k^\star$, so we can consider profile $P \in \mathcal{P}^n$ such that $P_2 : \ldots, b, a, \ldots$ where b is in the $k^\star$-th position and $P_j : a, b, \ldots$ for each $j \in N \setminus \{1\}$ and for each alternative above b in the preference of agent 2 there is another agent that has that alternative below the $k^\star$-th position. In this profile, as $s_{k^\star-1} = s_{k^\star}$, alternative a has a score greater than or equal to the score of any alternative above a for agent 2. Then, by the tie-breaking, the rule selects something worse than b for agent 2. Then, agent 2 can manipulate the rule by misrepresenting his preferences by interchanging the alternative in the $(k^\star+1)$-th position and b, because now b is a scoring winner, a is not, and thus the outcome is b. Furthermore, for any profile where agent 2 declares his true preference and the outcome of the rule is $f(P)$ (that is below b for agent 2), if agent 2 misrepresents interchanging as said previously, then the outcome of the rule is at least as good as $f(P)$. Hence, agent 2 regrets truth-telling.

The previous theorem extends the result of Theorem 3 in Reijngoud and Endriss (2012) to the case $n = 3$ and also to the neutral scoring rules (their result only applies when $n > 3$ in the anonymous case). Our proof is independent of theirs.

Corollary 4. *Assume $n > 2$ and $k^\star > 1$. Then,*

(i) An A-scoring rule with $s_{k^\star-1} = s_{k^\star}$ is regret-free truth-telling if and only if $k^\star n = m - 1$. In particular, an anonymous $(m - k^\star)$-approval rule is regret-free truth-telling if and only if $k^\star n = m - 1$.

(ii) An N-scoring rule with $s_{k^\star-1} = s_{k^\star}$ is regret-free truth-telling if and only if $k^\star n < m$. In particular, a neutral $(m - k^\star)$-approval rule is regret-free truth-telling if and only if $k^\star n < m$.

Corollary 4 (i) contradicts Theorem 2 in Endriss et al. (2016) which states that there is no anonymous approval rule satisfying regret-free truth-telling. The proof in Endriss et al. (2016) assumes that there is a preference profile in which the outcome is below the position $k^\star + 1$ for some agent, however, this assumption cannot be met in the case where $k^\star n < m$.

4.3 Condorcet consistent rules

Let $P \in \mathcal{P}^n$ and consider two alternatives $a, b \in X$. Denote by $C_P(a, b)$ the number of agents that prefer a to b according to P, i.e., $C_P(a, b) = |\{i \in N : aP_ib\}|$. An alternative $a \in X$ is a *Condorcet winner* according to P if for each alternative $b \in X \setminus \{a\}$,

$$C_P(a, b) > C_P(b, a). \tag{5}$$

Notice that a Condorcet winner may not always exist but when it does, it is unique. If (5) holds with weak inequality for each alternative $b \in X \setminus \{a\}$, then a is called a *weak Condorcet winner.*

Definition 5. *A rule $f : \mathcal{P}^n \to X$ is **Condorcet consistent** if it chooses the Condorcet winner whenever it exists.*

Next, we introduce a mild monotonicity condition which says that if an alternative is below the outcome for an agent and he changes his preferences modifying only the ordering of alternatives above the outcome, then such alternative continues not to be chosen. Formally,

Definition 6. *Let $P_i, P_i' \in \mathcal{P}$ and let $a \in X$ be such that $a = t_k(P_i)$. We say that P_i' is a **monotonic transformation of P_i with respect to a** if $t_{k'}(P_i) = t_{k'}(P_i')$ for each $k' \leq k$. A rule $f : \mathcal{P}^n \to X$ is **monotone** if, for each $P \in \mathcal{P}^n$, each $i \in N$, and each $b \in X$ such that $f(P)P_ib$,*

$$f(P_i', P_{-i}) \neq b$$

for each P_i' that is a monotonic transformation of P_i with respect to $f(P)$.

Notice that $C_P(x, b) = C_{(P_i', P_{-i})}(x, b)$ for each $x \in X \setminus \{b\}$. Thus, our monotonicity condition is fully compatible with Condorcet consistency.

Furthermore, our notion of monotonicity is weaker than the well-known Maskin monotonicity. Remember that $P_i' \in \mathcal{P}$ is a *Maskin monotonic transformation* of $P_i \in \mathcal{P}$ with respect to $a \in X$ if $xP_i'a$ implies xP_ia. Then, $f : \mathcal{P}^n \to X$ is *Maskin monotonic* if, for each $P \in \mathcal{P}^n$, $f(P_i', P_{-i}) = f(P)$ for each $P_i' \in \mathcal{P}$ that is a Maskin monotonic transformation of P_i with respect to $f(P)$. It is clear that a monotonic transformation of P_i (according to our definition) is a Maskin monotonic transformation of P_i.

Besides the intrinsic appeal of our monotonicity condition, this weakening of Maskin's property is necessary since Maskin's monotonicity is incompatible with Condorcet consistency. To see this, let $X = \{a, b, c\}$ and consider a Condorcet consistent $f : \mathcal{P}^3 \to X$ and a profile $P \in \mathcal{P}^3$ given by the following table:

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b

Since there is no Condorcet winner, and without loss of generality, assume $f(P) = a$. Now, let $P_1' \in \mathcal{P}$ be such that $cP_1'bP_1'a$. It follows, by Condorcet consistency, that $f(P_1', P_{-1}) = c$. If f is also Maskin monotonic, then $f(P_1', P_{-1}) = c$ implies $f(P) = c$, a contradiction.

Our mild monotonicity requirement on Condorcet consistent rules leads to the following negative result concerning regret-free truth-telling.

Theorem 6. *Assume $n \notin \{2, 4\}$, or $n = 4$ and $m > 3$. Then, there is no Condorcet consistent, monotone and regret-free truth-telling rule.*

Proof. See Appendix A.6. □

The main ideas behind the proof of Theorem 6 can be illustrated in the three-agent case, sketched next. Consider profile $P \in \mathcal{P}^3$ given by the following table:

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b
$\vdots$	$\vdots$	$\vdots$

In this profile there is no Condorcet winner. Let $f(P)$ be the chosen alternative under profile P. We can assume, w.l.o.g., that $f(P)$ is worse than b for agent 1. Now, agent 1 can manipulate the rule by interchanging a and b, because the outcome of the rule then changes to b. Furthermore, for any profile where agent 1 declares his true preference and the outcome of the rule is $f(P)$ (that is bellow b for agent 1), if agent 1 misrepresents his preferences by interchanging a and b, then by monotonicity the outcome of the rule is at least as good as $f(P)$. Hence, agent 1 regrets truth-telling.

Remark 3. *When $n = 4$ and $m = 3$, the previous impossibility result does not apply. Let $X = \{a, b, c\}$. Given $P \in \mathcal{P}^n$ and $x \in X$, let $bottom(P, x)$ be the number of agents that have x in the bottom of their preferences. Now, consider a rule $f : \mathcal{P}^4 \to X$ that, for each $P \in \mathcal{P}^4$, selects the Condorcet winner when it exists and, otherwise, the tie-breaking $a \succ b \succ c$ is used to choose an alternative among those that minimize $bottom(P, \cdot)$ and are preferred by at least two agents to any other alternative that minimizes $bottom(P, \cdot)$. This rule is monotone since, given $P \in \mathcal{P}^4$ and $i \in N$, when there are three alternatives a monotonic transformation of P_i with respect to $f(P)$ is different from P_i only when $t_1(P_i) = f(P)$, and in this case there is no $x \in X$ such that $f(P)P_i x$. Then, monotonicity is trivially satisfied. To see that this rule is also regret-free truth-telling, consider profile $P \in \mathcal{P}^4$ such that, w.l.o.g., $P_1 : x, y, z$. If $f(P) = x$, agent 1 does not manipulate f. If $f(P) = z$, then $f(P'_1, P_{-1}) = t_1(P_1)$ for each $P'_1 \in \mathcal{P}$ by definition of the rule, so agent 1 cannot manipulate either. If $f(P) = y$ and agent 1 manipulates f via P'_1, then $f(P'_1, P_{-1}) = x$ and, by definition of the rule, $t_1(P'_1) = y$. Consider $P^\star_{-1} \in \mathcal{P}^3$ such that $P^\star_2 : y, z, x$, $P^\star_3 = P^\star_2$, and $P^\star_4 : z, y, x$. Then, $f(P_1, P^\star_{-1}) = y$ and $f(P'_1, P^\star_{-1}) = z$. Therefore, agent 1 does not regret truth-telling.*

Six of the most important Condorcet consistent rules are Simpson, Copeland, Young, Dodgson, Fishburn and Black rules (see Fishburn, 1977). Each one of these rules uses pairwise comparison of alternatives in a specific way in order to get a *winner* alternative for each profile of preferences. Their definitions are as follows. Given $P \in \mathcal{P}^n$,

(i) the *Simpson score* of alternative $a \in X$ is the minimum number $C_P(a, b)$ for $b \neq a$,

$$Simpson(P, a) = \min_{b \neq a} C_P(a, b)$$

and a *Simpson winner* is an alternative with highest such score.

(ii) the *Copeland score* of alternative $a \in X$ is the number of pairwise victories minus the number of pairwise defeats against all other alternatives

$$Copeland(P, a) = |\{b : C_P(a, b) > C_P(b, a)\}| - |\{b : C_P(b, a) > C_P(a, b)\}|$$

and a *Copeland winner* is an alternative with highest such score.

(iii) the *Young score* of alternative $a \in X$ is the largest cardinality of a subset of voters for which alternative a is a weak Condorcet winner

$$Young(P, a) = \max_{N' \subseteq N} \left\{ |N'| : \{|i \in N' : aP_i b\}| \geq \frac{|N'|}{2} \text{ for all } b \in X \setminus \{a\} \right\}$$

and a *Young winner* is an alternative with highest such score.

(iv) the *Dodgson score* of alternative $a \in X$, $Dodgson(P, a)$, is the fewest inversions[8] in the preferences in P that will make a tie or beat every other alternative in X on the basis of simple majority, and a *Dodgson winner* is an alternative with lowest such score.

(v) the *Fishburn partial order* on X, F_P, is defined as follows: aF_Pb if and only if for each $x \in X$, $C_P(x, a) > C_P(a, x)$ implies $C_P(x, b) > C_P(b, x)$ and there is $w \in X$ such that $C_P(w, b) > C_P(b, w)$ and $C_P(a, w) \geq C_P(w, a)$. A *Fishburn winner* is a maximal alternative for F_P.

(vi) a *Black winner* is a Condorcet winner whenever it exists and, otherwise, a *Borda winner*.[9]

An *anonymous (neutral) Simpson, (Copeland, Young, Dodgson, Fishburn, Black) rule* always chooses a Simpson, (Copeland, Young, Dodgson, Fishburn, Black) winner and uses a fixed order (agent) as tie-breaker when there are more than one. The following result shows that the six rules are monotonic.

Corollary 5. *Assume $n > 2$. Then, the Simpson, Copeland, Young, Dodgson, Fishburn and Black rules are not regret-free truth-telling, regardless of whether we consider their anonymous or neutral versions.*

Proof. See Appendix A.7. □

Another interesting class of Condorcet consistent rules which are widely used in practice, for instance, by the United States Congress to vote upon a motion and its proposed amendments, is the class of succesive elimination rules (see Chapter 9 of Moulin, 1991, for more detail). These rules, which consider an order among alternatives and consist of sequential majority comparisons, are defined as follows.

Definition 7. *A rule $f : \mathcal{P}^n \to X$ is a **successive elimination** rule with respect to an order $\succ$ such that $a_1 \succ a_2 \succ \ldots \succ a_m$ if it operates in the following way. First, a majority vote decides to eliminate a_1 or a_2, then a majority vote decides to eliminate the survivor from the first round or a_3, and so on. The same order $\succ$ is used as tie-breaker in each pairwise comparison, if necessary.*

It is clear that a successive elimination rule is Condorcet consistent but it may be not monotone, as the next example shows.

Example 1. *(The successive elimination rule with respect to order $a \succ b \succ c \succ d$ is not monotone). Let $P \in \mathcal{P}^5$ be given by the following table:*

P_1	P_2	P_3	P_4	P_5
a	a	c	c	d
b	c	d	b	b
d	d	a	d	a
c	b	b	a	c

[8] Let $P_i, P'_i \in \mathcal{P}$ and let $x, y \in X$. P'_i is an *inversion of P_i with respect x and y* if xP_iy implies yP'_ix.

[9] A *Borda winner* is an alternative with highest Borda score.

Then, $f(P) = d$. Now, let $P'_1 \in \mathcal{P}$ be such that $P'_1 : b, a, d, c$. Then P'_1 is a monotonic transformation of P_1 with respect to d but $f(P'_1, P_{-1}) = c$, so f is not monotone.

Theorem 7. *Assume $n > 2$. Then, no successive elimination rule is regret-free truth-telling.*

Proof. See Appendix A.8. □

The main ideas behind the proof of Theorem 7 can be found in the three-agent case, so we sketch it here. Let f be a successive elimination rule with associated order $a \succ b \succ c \succ \dots$ Consider profile $P \in \mathcal{P}^3$ given by the following table:

P_1	P_2	P_3
a	b	c
b	c	a
$\vdots$	a	b
c	$\vdots$	$\vdots$

In this profile, the outcome of the rule is c. Now, agent 1 can manipulate the rule by interchanging a and b, because the outcome of the rule then changes to b. Furthermore, as c is the worst alternative in the true preference of agent 1, agent 1 regrets truth-telling.

5 Two agents and three alternatives: characterizations

In what follows, we focus in the case where we have only two agents, $N = \{1, 2\}$, and three alternatives, $X = \{a, b, c\}$. In this case we can obtain characterizations of the classes of all: (i) regret-free truth-telling and neutral, and (ii) regret-free truth-telling, efficient, and anonymous rules. Notice that for the first characterization efficiency is not needed since it is implied by neutrality and regret-free truth-telling, as we prove next in Theorem 8.

First, observe that with two agents and three alternatives a N-maxmin rule coincides with:

(i) the N- negative plurality rule; and,

(ii) the N-scoring rule corresponding to $\bar{s} = (\bar{s}_1, \bar{s}_2, \bar{s}_3) = (1, 3, 4)$.

The following theorem shows that N-maxmin and dictatorships are the only regret-free truth-telling and neutral rules.

Theorem 8. *Assume $n = 2$ and $m = 3$. Then, a rule is regret-free truth-telling and neutral if and only if it is a N-maxmin rule or a dictatorship.*

Proof. See Appendix A.9. □

A similar result to the previous theorem can be obtained changing neutrality for anonymity. As efficiency is not a consequence of anonymity and regret-free truth-telling we require it in the next theorem.[10]

[10] For example, a constant rule is regret-free truth-telling and anonymous but not efficient.

In this case, we need to enlarge the class of A-maxmin rules by dropping the requirement of transitivity for the tie-breaking associated to the rules and to add the successive elimination rules into the picture, as we did with dictatorial rules in Theorem 8.

Definition 8. *A rule $f : \mathcal{P}^2 \to X$ is an **A-maxmin*** rule if there is an antisymmetric and complete (not necessarily transitive) binary relation $\succ^\star$ on X such that, for each $P \in \mathcal{P}^2$,*

$$f(P) = \max_{\succ^\star} \mathcal{M}(P).$$

Observe that, since $n = 2$, $|\mathcal{M}(P)| \leq 2$ and therefore $\max_{\succ^\star} \mathcal{M}(P)$ is well defined. In a similar way to Definition 8 we can define the A-scoring* rule associated to $\succ^\star$. Notice that the A-maxmin* rule associated to $\succ^\star$ coincides with the A-scoring* rule with $\bar{s} = (\bar{s}_1, \bar{s}_2, \bar{s}_3) = (1, 3, 4)$ associated to $\succ^\star$.

Theorem 9. *Assume $n = 2$ and $m = 3$. Then, a rule is regret-free truth-telling, efficient, and anonymous if and only if it is a successive elimination rule or an A-maxmin* rule.*

Proof. See Appendix A.10. □

Concerning the independence of axioms in the characterizations, it is clear that neutrality and regret-free truth-telling in Theorem 8 are independent. Successive elimination rules are regret-free truth-telling but not neutral, and the rule that always chooses the bottom of agent 1 is neutral and not regret-free truth-telling. On the other hand, in Theorem 9, a constant rule is regret-free truth-telling, anonymous, and not efficient and a dictatorship is regret-free truth-telling, efficient, and not anonymous. Now, given order $a \succ b \succ c$, consider the rule $f(P) = \max_{\succ}\{t(P_1), t(P_2)\}$. This rule is anonymous, efficient, and not regret-free truth-telling.

$m = 2$	regret-free $\iff$ ext. majority voting	Cor. 1
$n = 2, m = 3$	regret-free + neutral $\iff$ N-maxmin or dictatorship	Th. 8
$n = 2, m = 3$	regret-free + eff. + anon. $\iff$ A-maxmin* or succ. elim.	Th. 9

We use "regret-free" to mean "regret-free truth-telling" due to space considerations.

Table 1: *Characterization results with $m = 2$ or $n = 2$ and $m = 3$.*

6 Summary

Table 1 summarizes the characterization results when there are only two alternatives, or two agents and three alternatives. Table 2 summarizes our main findings about tops-only, maxmin, scoring, and Condorcet consistent rules.

Tops-only			strategy-proof $\iff$ regret-free	Pr. 1
A-maxmin			$n \geq m-1$ or n divides $m-1$ $\iff$ regret-free	Th. 1
N-maxmin			all regret-free	Th. 1
A-scoring[†] ($n > 2$)	$k^\star = 1$		$n \geq m-1$ $\iff$ regret-free	Th. 2
	$1 < k^\star < m-1$	$k^\star n < m$	$k^\star n = m-1$ $\iff$ regret-free	Th. 4
		$k^\star n \geq m$	$s_{k^\star-1} = s_{k^\star}$ $\implies$ none regret-free	Th. 5
	$k^\star = m-1$		none regret-free	Th. 3
N-scoring[†] ($n > 2$)	$k^\star = 1$		all regret-free	Th. 2
	$1 < k^\star < m-1$	$k^\star n < m$	all regret-free	Th. 4
		$k^\star n \geq m$	$s_{k^\star-1} = s_{k^\star}$ $\implies$ none regret-free	Th. 5
	$k^\star = m-1$		none regret-free	Th. 3
Condorcet consistent ($n > 2$)	Monotone		$n \neq 4$ or $m > 3$ $\implies$ none regret-free	Th. 6
	Succesive elimination		none regret-free	Th. 7

[†] Remember that $k^\star$ is such that $s_1 \leq s_2 \leq s_{k^\star} < s_{k^\star+1} = \ldots = s_m$. The results for $k^\star = 1$ also apply when $n = 2$.
We use "regret-free" to mean "regret-free truth-telling" due to space considerations.

Table 2: *Summary of results for tops-only, maxmin, scoring, and Condorcet consistent rules.*

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A Appendix

A.1 Proof of Theorem 1

We first show the equivalence in part (i). Let $f : \mathcal{P}^n \to X$ be a A-maxmin rule.
($\Longrightarrow$) Assume that $n < m-1$ and that n does not divide $m-1$. Then, there are $h \geq 1$ and $1 \leq s < n$ such that $m-1 = nh+s$, or $m = nh+r$ with $h \geq 1$ and $2 \leq r \leq n$. As $m - nh = r$, for any profile of preferences there are at least r alternatives whose minimal position is at least $h+1$. So, the minimal position of a maxmin winner is always at least $h+1$, and in the case that it is exactly $h+1$, there are at least r maxmin winners. Let $P \in \mathcal{P}^n$ be given by the following table:

	P_1	P_2	P_3	$\dots$	P_r	P_{r+1}	$\dots$	P_n
	b	a	b	$\dots$	b	b	$\dots$	b
	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
	a	b	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
last h positions	z	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\vdots$

where each alternative appears exactly one time below the dashed line (this can be done because $m = nh + r$).

Then, $a, b \in \mathcal{M}(P)$ and $f(P) = a$. Now, consider preference $P_1' \in \mathcal{P}$ that differs from P_1 only in that the positions of a and z are interchanged. We have that $a \notin \mathcal{M}(P_1', P_{-1})$, $b \in \mathcal{M}(P_1', P_{-1})$, and $f(P_1', P_{-1}) = b$. Therefore,

$$f(P_1', P_{-1}) P_1 f(P_1, P_{-1}). \tag{6}$$

Let $P^\star_{-1} \in \mathcal{P}^{n-1}$ be such that $f(P_1, P^\star_{-1}) = f(P) = a$. As we noted in the first paragraph of this proof, it follows that $mp(f(P_1', P^\star_{-1}), (P_1', P^\star_{-1})) \geq h + 1$. There are two cases to consider:

1. $\boldsymbol{mp(f(P_1', P^\star_{-1}), (P_1', P^\star_{-1})) > h + 1}$**.** Then, $f(P_1', P^\star_{-1}) P_1' z$ and, by the definition of P_1',
$$f(P_1', P^\star_{-1}) P_1 f(P_1, P^\star_{-1}). \tag{7}$$
By (6) and (7), f is not regret-free truth-telling.

2. $\boldsymbol{mp(f(P_1', P^\star_{-1}), (P_1', P^\star_{-1})) = h + 1}$**.** As we noted in the first paragraph of this proof, $\left|\mathcal{M}(P_1', P^\star_{-1})\right| \geq r \geq 2$ and $f(P_1', P^\star_{-1}) \neq z$ (because z is the last one in order $\succ$). Again, $f(P_1', P^\star_{-1}) P_1' z$ and the proof follows as in the previous case.

($\Longleftarrow$) Assume that there exist $i \in N$, $(P_i, P_{-i}) \in \mathcal{P}^n$ and $P_i' \in \mathcal{P}$ such that

$$f(P_i', P_{-i}) P_i f(P_i, P_{-i}). \tag{8}$$

We will prove that there is $P^\star_{-i} \in \mathcal{P}^{n-1}$ such that $f(P) = f(P_i, P^\star_{-i})$ and $f(P_i, P^\star_{-i}) P_i f(P_i', P^\star_{-i})$. Let $\widehat{P} = (P_i', P_{-i})$. As f is an A-maxmin rule,

$$mp(P, f(P)) \geq mp(P, f(\widehat{P})) \tag{9}$$

and

$$mp(\widehat{P}, f(\widehat{P})) \geq mp(\widehat{P}, f(P)). \tag{10}$$

Let $\overline{k}$ be such that $t_{\overline{k}}(P_i) = f(P)$. By (9) and since $f(\widehat{P}) P_i f(P)$,

$$mp(P, f(\widehat{P})) = k^\star \leq \overline{k} \leq m - 1, \tag{11}$$

where $t_{k^\star}(\widehat{P}_j) = f(\widehat{P})$ for some $j \in N \setminus \{i\}$. Then, as $\widehat{P}_j = P_j$ and $k^\star \leq \overline{k}$,

$$mp(P, f(\widehat{P})) \geq mp(\widehat{P}, f(\widehat{P})). \tag{12}$$

If $mp(P, f(P)) = mp(P, f(\widehat{P}))$ and $mp(\widehat{P}, f(\widehat{P})) = mp(\widehat{P}, f(P))$, then

$$f(P), f(\widehat{P}) \in \mathcal{M}(\widehat{P}) \cap \mathcal{M}(P), \tag{13}$$

contradicting that $f(P) \neq f(\widehat{P})$. Therefore, by (9) and (10), $mp(\widehat{P}, f(\widehat{P})) > mp(\widehat{P}, f(P))$ or $mp(P, f(P)) > mp(P, f(\widehat{P}))$. By (12),

$$mp(P, f(P)) > mp(\widehat{P}, f(P)). \tag{14}$$

Let $\widehat{k}$ be such that $mp(\widehat{P}, f(P)) = \widehat{k}$. Then, by (14), $t_{\widehat{k}}(\widehat{P}_i) = f(P)$ and $f(P)\widehat{P}_j t_{\widehat{k}}(\widehat{P}_j)$ for all $j \in N \setminus \{i\}$. If $\bar{k} \leq \widehat{k}$, then

$$mp(\widehat{P}, f(P)) = \widehat{k} \geq \bar{k} \geq mp(P, f(P)),$$

which contradicts (14). Therefore,

$$\bar{k} > \widehat{k}. \tag{15}$$

This implies that there exists an alternative $x \in X$ such that

$$f(P) = t_{\bar{k}}(P_i)P_i x \text{ and } xR_i' t_{\bar{k}}(P_i')P_i' f(P). \tag{16}$$

There are two cases to consider:

1. $\boldsymbol{n \geq m-1}$. Let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P^\star_j) = f(P)$, $t_{m-1}(P^\star_j) = x$ for each $j \in N \setminus \{i\}$, and for each $x \in X \setminus \{f(P), x\}$ choose an agent j^x such that $t_1(P^\star_{j^x}) = x$ (this is feasible because $n-1 \geq m-2$). Now, let $P^\star = (P_i, P^\star_{-i})$. Then, $mp((P_i, P^\star_{-i}), y) = 1$ for all $y \in X \setminus \{f(P), x\}$ and from definition of $P^\star_{-i}$, (16) and the fact that $\bar{k} \leq m-1$, we have $mp((P_i, P^\star_{-i}), f(P)) = \bar{k} > mp((P_i, P^\star_{-i}), x)$. Therefore, $f(P_i, P^\star_{-i}) = f(P)$. Furthermore, $mp((P_i', P^\star_{-i}), y) = 1$ for each $y \in X \setminus \{f(P), x\}$ and from definition of $P^\star_{-i}$, (16) and the fact that $\bar{k} \leq m-1$, we have $mp((P_i', P^\star_{-i}), x) > mp((P_i', P^\star_{-i}), f(P))$. Therefore, $f(P_i', P^\star_{-i}) = c$.

   We conclude that $f(P) = f(P_i, P^\star_{-i})$ and, by (16) and the fact that $f(P_i', P^\star_{-i}) = x$, $f(P_i, P^\star_{-i})P_i f(P_i', P^\star_{-i})$. Hence, f is regret-free truth-telling.

2. $\boldsymbol{n}$ **divides** $\boldsymbol{m-1}$. Thus, $m-1 = hn$ with $h \geq 1$. Therefore,

$$mp(P, f(P)) \geq h+1. \tag{17}$$

   Let $Y = \{y \in X : yP_i f(P)\}$. Then,

$$|Y| < m - mp(P, f(P)) \leq m - (h+1) = hn + 1 - h - 1 = h(n-1). \tag{18}$$

   Let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P^\star_j) = f(P)$, $t_{m-1}(P^\star_j) = x$ for each $j \in N \setminus \{i\}$, and for each $y \in Y$ choose an agent j and a position $u \leq h$ such that $t_u(P^\star_j) = y$ (the construction of $P^\star_{-i}$ is feasible by (18) and the fact that $m-2 = hn-1 \geq h(n-1)$). Now, let $P^\star = (P_i, P^\star_{-i})$. Then, $mp((P_i, P^\star_{-i}), y) \leq h$ for each $y \in Y$, $mp((P_i, P^\star_{-i}), f(P)) \geq mp(P, f(P)) \geq h+1$ (this holds by (17) and the definition of $P^\star_{-i}$), and $mp((P_i, P^\star_{-i}), f(P)) = \bar{k} > mp((P_i, P^\star_{-i}), y)$ for each $y \in X \setminus Y$ (this follows from the definitions of $P^\star_{-i}$ and Y). Hence, $f(P_i, P^\star_{-i}) = f(P)$. Furthermore, $mp((P_i', P^\star_{-i}), y) \leq h$ for each $y \in Y$ and $mp((P_i', P^\star_{-i}), x) > mp((P_i', P^\star_{-i}), f(P))$ (this follows from (11), (16), and the definition of $P^\star_{-i}$). Therefore, $f(P_i', P^\star_{-i}) \in X \setminus Y$ and $f(P_i', P^\star_{-i}) \neq f(P)$.

   We conclude that $f(P) = f(P_i, P^\star_{-i})$ and $f(P_i, P^\star_{-i})P_i f(P_i', P^\star_{-i})$. Hence, f is regret-free truth-telling.

Next, we show part (ii). Assume that $f : \mathcal{P}^n \to X$ is a N-maxmin rule. Then, there exists $\bar{j} \in N$ such that

$$f(\widetilde{P}) = \max_{\widetilde{P}_{\bar{j}}} \mathcal{M}(\widetilde{P}) \text{ for each } \widetilde{P} \in \mathcal{P}^n. \tag{19}$$

Let $P, P_i', \widehat{P}, \bar{k}$, and $\widehat{k}$ be as in ($\Longleftarrow$) of part (i). It is easy to see that equations (8), (9), (10), (11) and (12) also hold here.

If $mp(P, f(P)) = mp(P, f(\widehat{P}))$ and $mp(\widehat{P}, f(\widehat{P})) = mp(\widehat{P}, f(P))$, then (13) holds as in the proof of part (i). As $f(P_i', P_{-i})P_i f(P_i, P_{-i})$, we have $\bar{j} \neq i$. But then (13) contradicts $f(P) \neq f(\widehat{P})$ since $P_{\bar{j}} = \widehat{P}_{\bar{j}}$. Therefore, by (9) and (10), $mp(\widehat{P}, f(\widehat{P})) > mp(\widehat{P}, f(P))$ or $mp(P, f(P)) > mp(P, f(\widehat{P}))$. Now, it is easy to see that equations (14), (15) and (16) hold in this proof as well, so there exists $x \in X$ such that $f(P) = t_{\bar{k}}(P_i)P_i x$ and $xR_i' t_{\bar{k}}(P_i')P_i' f(P)$.

Now, we define profile $P_{-i}^\star \in \mathcal{P}^{n-1}$ where, for each $j \in N \setminus \{i\}$, $P_j^\star$ is differs from P_j in that $f(P)$ is now in the top of $P_j^\star$ and x is in the second place, while all the other alternatives keep their relative ranking. Formally, let $P_{-i}^\star \in \mathcal{P}^{n-1}$ be such that, for each $j \in N \setminus \{i\}$, $t(P_j^\star) = f(P)$, $t_{m-1}(P_j^\star) = x$, and if k' and k'' are such that $t_{k'}(P_j) = f(P)$ and $t_{k''}(P_j) = x$, if we let $k_1 = \max\{k', k''\}$ and $k_2 = \min\{k', k''\}$, define

$$t_k(P_j^\star) = \begin{cases} t_{k+2}(P_j) & \text{if } m-2 \geq k \geq k_1 - 1, \\ t_{k+1}(P_j) & \text{if } \bar{k} - 1 > k \geq k_2. \end{cases}$$

Next, we present two claims.

Claim 1: $f(P_i, P_{-i}^\star) = f(P)$. Let $P^\star = (P_i, P_{-i}^\star)$. Since $f(P)P_i x$ and by definition of $P_{-i}^\star$,

$$mp(P^\star, f(P)) > mp(P^\star, x) \tag{20}$$

Then, $f(P^\star) \neq x$. As $f(P_i', P_{-i})P_i f(P) = t_{\bar{k}}(P_i)$,

$$mp(P^\star, f(P)) = \bar{k}. \tag{21}$$

Now, let $b \in X \setminus \{f(P), x\}$. By definition of $P^\star$ and the fact that f is a N-maxmin rule,

$$mp(P^\star, b) \leq mp(P, b) \leq mp(P, f(P)) \leq \bar{k}. \tag{22}$$

Therefore, $f(P) \in \mathcal{M}(P^\star)$ and $mp(P^\star, f(P)) = \bar{k}$. On the one hand, if $\bar{j} \neq i$, then $t(P_{\bar{j}}^\star) = f(P)$ and, by definition of f, $f(P) = f(P^\star)$. On the other hand, if $\bar{j} = i$ and there is $b \in \mathcal{M}(P^\star) \setminus \{f(P)\}$, then by (42) and (22), $mp(P, b) = mp(P, f(P)) = \bar{k}$. Thus, by (19) and the fact that $\bar{j} = i$, $f(P)P_i b$. Therefore, as $P_i^\star = P_i$,$f(P) = f(P^\star)$. This proves the Claim.

Claim 2: $f(P_i, P_{-i}^\star)P_i f(P_i', P_{-i}^\star)$. If $f(P_i', P_{-i}^\star) = x$, then by Claim 1 and (16) the proof is trivial. Now assume $f(P_i', P_{-i}^\star) \neq x$. First, we will prove that $f(P_i', P_{-i}^\star) \neq f(P)$. As $f(P) = t_{\widehat{k}}(P_i')$,

$$\widehat{k} = mp((P_i', P_{-i}^\star), f(P)).$$

Furthermore, as $xR_i' t_{\bar{k}}(P_i')$ and $\bar{k} \leq m-1$, by definition of $P^\star_{-i}$,

$$mp((P_i', P^\star_{-i}), x) \geq \bar{k}. \tag{23}$$

Then, by (15),

$$mp((P_i', P^\star_{-i}), x) > \widehat{k} = mp((P_i', P^\star_{-i}), f(P)),$$

implying that $f(P_i', P^\star_{-i}) \neq f(P)$.

Now, let $b \in X \setminus \{f(P), x\}$ be such that $bP_i f(P_i, P^\star_{-i})$. Since $f(P_i, P^\star_{-i}) = f(P) = t_{\bar{k}}(P_i)$, by definition of f there exists $j \in N \setminus \{i\}$ such that $t_{\bar{k}}(P_j)R_j b$. By definition of $P^\star_{-i}$, $t_{\bar{k}}(P^\star_j)R^\star_j b$. Therefore,

$$mp((P_i', P^\star_{-i}), b) \leq \bar{k}. \tag{24}$$

On the one hand, if $\bar{j} \neq i$, since $t_{m-1}(P^\star_{\bar{j}}) = x$ the definition of f, (23), and (24) imply that $f(P_i', P^\star_{-i}) \neq b$. On the other hand, if $\bar{j} = i$, since $bP_i f(P_i, P^\star_{-i})$ the definition of f implies $mp((P_i, P^\star_{-i}), b) < mp((P_i, P^\star_{-i}), f((P_i, P^\star_{-i})))$. Then,

$$mp((P_i, P^\star_{-i}), b) < \bar{k}.$$

Therefore, as $bP_i f(P_i, P^\star_{-i}) = t_{\bar{k}}(P_i)$,

$$mp((P_i', P^\star_{-i}), b) < \bar{k}$$

Then, by the definition of f and (23), $f(P_i', P^\star_{-i}) \neq b$ in this case as well. Therefore, we conclude that

$$f(P_i, P^\star_{-i}) P_i f(P_i', P^\star_{-i}),$$

proving the Claim.

By Claims 1 and 2 we conclude that f is regret-free truth-telling. □

A.2 Proof of Theorem 2

We first show the equivalence in part (i). Let $f : \mathcal{P}^n \to X$ be an A-scoring rule with $k^\star = 1$. ($\Longrightarrow$) Suppose that $n < m-1$ (this implies $m > 3$). Assume that a, b are the first two alternatives in the tie-breaking with $a \succ b$ and let z the last alternative in the tie-breaking. Let $P \in \mathcal{P}^n$ be such that $t_3(P_i) = b$, $t_2(P_i) = a$, $t_1(P_i) = z$, and $t_m(P_j) = b$, $t_{m-1}(P_j) = a$, and $t_{m-2}(P_j) = z$ for each $j \in N \setminus \{i\}$. Then, $f(P) = a$. Now, let $P_i' \in \mathcal{P}$ be such that $t_1(P_i') = a$. Then, $f(P_i', P_{-i}) = b$ and, therefore,

$$f(P_i', P_{-i}) P_i f(P). \tag{25}$$

Now, let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $f(P) = f(P_i, P^\star_{-i})$. As $n+1 < m$, $\left|\mathcal{S}(P_i', P^\star_{-i})\right| \geq 2$. Therefore, as z is the last alternative in the order $\succ$, $f(P_i', P^\star_{-i}) \neq z$ and

$$f(P_i', P^\star_{-i}) R_i f(P_i, P^\star_{-i}). \tag{26}$$

Hence, by (25) and (26), f is not regret-free truth-telling.

($\Longleftarrow$) Assume $n \geq m-1$ and there exist $i \in N$, $P \in \mathcal{P}^n$ and $P_i' \in \mathcal{P}$ such that

$$f(P_i', P_{-i}) P_i f(P). \tag{27}$$

Next, we show there is $P^\star_{-i} \in \mathcal{P}^{n-1}$ such that $f(P) = f(P_i, P^\star_{-i})$ and $f(P_i, P^\star_{-i}) P_i f(P_i', P^\star_{-i})$. Let $\widehat{P} = (P_i', P_{-i})$. If $t_1(P_i) = t_1(P_i')$, then $\mathcal{S}(P) = \mathcal{S}(\widehat{P})$, contradicting the definition of f and the fact that $f(\widehat{P}) \neq f(P_i, P_{-i})$. Therefore,

$$t_1(P_i) \neq t_1(P_i'). \tag{28}$$

If $t_1(P_i) = f(P)$, then as $t_1(P_i) \neq t_1(P_i')$, $s(\widehat{P}, f(P)) > s(P, f(P))$. By definition of f, $s(P, f(P)) \geq s(P, x)$ for each $x \in X$. Then, $s(\widehat{P}, f(P)) > s(P, x)$ for each $x \in X \setminus \{f(P)\}$. Now, as $\widehat{P} = (P_i', P_{-i})$, $s(\widehat{P}, f(P)) > s(\widehat{P}, x)$ for each $x \in X \setminus \{t_1(P_i)\}$. Therefore, as $t_1(P_i) = f(P)$, $f(\widehat{P}) = f(P)$ which contradicts (27). Thus,

$$t_1(P_i) \neq f(P). \tag{29}$$

Furthermore,

$$\begin{aligned} s(\widehat{P}, x) &= s(P, x) \text{ for each } x \notin \{t_1(P_i), t_1(P_i')\}, \\ s(\widehat{P}, t_1(P_i)) &= s(P, t_1(P_i)) + 1, \text{and} \\ s(\widehat{P}, t_1(P_i')) &= s(P, t_1(P_i')) - 1. \end{aligned}$$

Then, as $s(P, x) \leq s(P, f(P))$ for each $x \in X$,

$$\mathcal{S}(\widehat{P}) = \{t_1(P_i)\} \text{ or } \mathcal{S}(P) \setminus \{t_1(P_i')\} \subset \mathcal{S}(\widehat{P}) \subset \mathcal{S}(P) \cup \{t_1(P_i)\}.$$

Thus, by (27),

$$\mathcal{S}(P) \setminus \{t_1(P_i')\} \subset \mathcal{S}(\widehat{P}) \subset \mathcal{S}(P) \cup \{t_1(P_i)\}. \tag{30}$$

Next, we claim that

$$t_1(P_i') = f(P) \tag{31}$$

holds. Assume otherwise that $t_1(P_i') \neq f(P)$. Then, by (30) and the definition of f, $f(\widehat{P}) = f(P)$ or $f(\widehat{P}) = t_1(P_i)$, which contradicts that $f(\widehat{P}) P_i f(P)$. Then, (31) holds.

Now, let $P^\star_{-i}$ be such that $\{t_1(P^\star_j) : j \neq i\} = X \setminus \{f(P), t_1(P_i)\}$ ($P^\star_{-i}$ exists because $n \geq m-1$). As $t_1(P_i) \neq f(P)$, $\mathcal{S}(P_i, P^\star_{-i}) = \{f(P)\}$ and, therefore,

$$f(P) = f(P_i, P^\star_{-i})$$

By (31), $t_1(P_i') = f(P)$. Then, $\mathcal{S}(P_i', P^\star_{-i}) = \{t_1(P_i)\}$ implying $f(P_i', P^\star_{-i}) = t_1(P_i)$ and, therefore,

$$f(P_i, P^\star_{-i}) P_i f(P_i', P^\star_{-i}). \tag{32}$$

By (27) and (32), f is regret-free truth-telling.

In order to see (ii), Assume that f is a N-scoring rule with $k^\star = 1$. Then, there exists $\bar{j}$ such that

$$f(\widetilde{P}) = \max_{\widetilde{P}_{\bar{j}}} \mathcal{S}(\widetilde{P}) \text{ for each } \widetilde{P} \in \mathcal{P}^n. \tag{33}$$

Let $P, P'_i, \widehat{P}$, be as in ($\Longleftarrow$) of part (i). By definition, (27) also holds here.

If $t_1(P_i) = t_1(P'_i)$, then $\mathcal{S}(P) = \mathcal{S}(\widehat{P})$. As $f(P'_i, P_{-i}) P_i f(P_i, P_{-i})$, we have $\bar{j} \in N \setminus \{i\}$. Now, $\mathcal{S}(P) = \mathcal{S}(\widehat{P})$ contradicts $f(P) \neq f(\widehat{P})$ since $P_{\bar{j}} = \widehat{P}_{\bar{j}}$. Therefore, (28) holds here and it follows that both (29) and (30) hold as well. If $\bar{j} = i$, we get a contradiction with $f(\widehat{P}) P_i f(P)$ and $\mathcal{S}(\widehat{P}) \subset \mathcal{S}(P) \cup \{t_1(P_i)\}$, so $\bar{j} \neq i$.

Now, let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P^\star_{\bar{j}}) = t_1(P_i)$, $t_{m-1}(P^\star_{\bar{j}}) = f(P)$ and, for each $j \in N \setminus \{i, \bar{j}\}$, $t(P^\star_j) = f(P)$ and $t_2(P^\star_j) = t_1(P_i)$. Therefore, by (29), $f(P) \in \mathcal{S}(P_i, P^\star_{-i})$ and $t_1(P_i) \notin \mathcal{S}(P_i, P^\star_{-i})$. By definition of f and $P^\star_{\bar{j}}$ it follows that

$$f(P) = f(P_i, P^\star_{-i}).$$

Then, by (28), $t_1(P_i) \in \mathcal{S}(P'_i, P^\star_{-i})$. By definition of f and $P^\star_{\bar{j}}$ we have

$$f(P'_i, P^\star_{-i}) = t_1(P_i)$$

Therefore,

$$f(P_i, P^\star_{-i}) P_i f(P'_i, P^\star_{-i}). \tag{34}$$

By (27) and (34), f is regret-free truth-telling. □

A.3 Proof of Theorem 3

Let $f : \mathcal{P}^n \to X$ be a scoring rule with $k^\star = m - 1$ (this implies that $s_{m-1} < s_m$). Let $a, b, c \in X$ and assume w.l.o.g. that if f is an A-scoring then the tie-breaking is given by order $\succ$ with $a \succ b \succ c \succ \ldots$, whereas if f is a N-scoring rule agent 1 break ties. There are two cases to consider:

1. $\boldsymbol{n = 2t + 3}$ **with** $\boldsymbol{t \geq 0}$. Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	P_2	P_3	P_4	$\cdots$	P_{t+3}	P_{t+4}	$\cdots$	P_{2t+3}
a	c	b	a	$\cdots$	a	b	$\cdots$	b
c	b	a	b	$\cdots$	b	a	$\cdots$	a
b	a	c	c	$\cdots$	c	c	$\cdots$	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
				t agents			t agents	

As $s(P, a) = s(P, b) \geq s(P, x)$ for each $x \in X \setminus \{a, b\}$, by the tie-breaking it follows that $f(P) = a$. Let $P'_2 \in \mathcal{P}$ be such that $P'_2 : b, c, a, \ldots$, and let $\widehat{P} = (P'_2, P_{-2})$. As $k^\star = m - 1$, we have that

$$s(\widehat{P}, b) > s(P, b) = s(P, a) = s(\widehat{P}, a)$$

and

$$s(\widehat{P}, b) > s(P, b) \geq s(P, c) > s(\widehat{P}, c).$$

Therefore,

$$f(\widehat{P}) = bP_2a = f(P). \tag{35}$$

Next, consider $P^\star_{-2} \in \mathcal{P}^{n-1}$ such that $f(P_2, P^\star_{-2}) = a$. Then, $f(P'_2, P^\star_{-2}) \in \{a, b\}$ because $s((P_2, P^\star_{-2}), b) < s((P'_2, P^\star_{-2}), b), s((P_2, P^\star_{-2}), c) > s((P'_2, P^\star_{-2}), c)$, and $s((P_2, P^\star_{-2}), x) = s((P'_2, P^\star_{-2}), x)$ for each $x \in X \setminus \{b, c\}$. Therefore,

$$f(P'_2, P^\star_{-2}) R_2 f(P_2, P^\star_{-2}) \tag{36}$$

By (35) and (36), f is not regret-free truth-telling.

2. **$n = 2t$ with $t \geq 2$.** Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	P_2	P_3	P_4	$\cdots$	P_{t+1}	P_{t+2}	$\cdots$	P_{2t}
a	c	c	a	$\cdots$	a	b	$\cdots$	b
b	b	a	b	$\cdots$	b	a	$\cdots$	a
c	a	b	c	$\cdots$	c	c	$\cdots$	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
			$t-2$ agents			$t-1$ agents		

Then, $f(P) \in \{a, b, c\}$. Furthermore, as $s(P, a) = s(P, b)$, $a \succ b$ and aP_1b, $f(P) \in \{a, c\}$. If $f(P) = a$, then, $s(P, a) = s(P, b) \geq s(P, c)$ and we proceed as in Case 1. If $f(P) = c$, then $s(P, c) \geq s(P, a) = s(P, b)$. Consider agent j such that $t + 2 \leq j \leq 2t$ (i.e., $P_j : b, a, c, \ldots$) and let $P'_j \in \mathcal{P}$ be such that $P'_j : a, b, c, \ldots$ and $\widehat{P} = (P'_j, P_{-j})$. As $k^\star = m - 1$, $s(\widehat{P}, a) \geq s(\widehat{P}, c)$ and $s(\widehat{P}, a) > s(\widehat{P}, b)$. Since $a \succ c$ and aP_1c,

$$f(\widehat{P}) = aP_jc = f(P). \tag{37}$$

Next, consider $P^\star_{-j} \in \mathcal{P}^{n-1}$ such that $f(P_j, P^\star_{-j}) = c$. Then, $f(P'_j, P^\star_{-j}) \in \{c, a\}$, because $s((P_j, P^\star_{-j}), a) < s((P'_j, P^\star_{-j}), a)$, $(s(P_j, P^\star_{-j}), b) > s((P'_j, P^\star_{-j}), b)$, and $s((P_j, P^\star_{-j}), x) = s((P'_j, P^\star_{-j}), x)$ for each $x \in X \setminus \{b, a\}$. Therefore,

$$f(P'_j, P^\star_{-j}) R_j f(P_j, P^\star_{-j}) \tag{38}$$

By (37) and (38), f is not regret-free truth-telling.

□

A.4 Proof of Theorem 4

Assume $n > 2$ and let $f : \mathcal{P}^n \to X$ be a scoring rule such that $k^\star n < m$. We first show the equivalence in part (i). Let further assume that f is an A-scoring rule.
($\Longrightarrow$) Assume that $k^\star n < m - 1$, we will prove that f is not regret-free truth-telling. Assume that a and b are the first two alternatives in the tie-breaking $\succ$ with $a \succ b$ and let z the last alternative in the tie-breaking. First, notice that for any profile of preferences

there are at least two alternatives above position $k^\star$ in the preference of each agent. So, there are at least two score winners (with score equal to ns_m).

Let $P \in \mathcal{P}^n$ be such that $t(P_i) = b$, $t_{k^\star+1}(P_i) = a$, $t_{k^\star}(P_i) = z$ and, for each $j \in N \setminus \{i\}$, $P_j : b, a, \ldots$. Then, $f(P) = a$. Now, consider preference $P'_1 \in \mathcal{P}$ that differs from P_1 only in that the positions of a and z are interchanged. Therefore, $f(P'_i, P_{-i}) = b$ and

$$f(P'_i, P_{-i}) P_i f(P). \tag{39}$$

Now, let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $f(P_i, P^\star_{-i}) = f(P)$. As we noted in the first paragraph of this proof, $s(f(P'_i, P^\star_{-i}), (P'_i, P^\star_{-i})) = ns_m$ and $f(P'_i, P^\star_{-i}) \neq z$ (because z is the last alternative in the tie-breaking). Then, $f(P'_i, P^\star_{-i}) P'_i z$ and, by the definition of P'_i,

$$f(P'_i, P^\star_{-i}) R_i f(P_i, P^\star_{-i}). \tag{40}$$

By (39) and (40), f is not regret-free truth-telling.

($\Longleftarrow$) Assume that $k^\star n = m - 1$. Then, for any profile there is always an alternative with maximal score $n \cdot s_m$. Thus, given $P \in \mathcal{P}^n$, $s(f(P), P) = n \cdot s_m$ and $f(P) P_j t_{k^\star}(P_j)$ for each $j \in N$.

Let $P'_i \in \mathcal{P}$ be such that

$$f(P'_i, P_{-i}) P_i f(P). \tag{41}$$

Then, $f(P'_i, P_{-i}) P_i f(P) P_i t_{k^\star}(P_i)$. By definition of $k^\star$,

$$s(f(P'_i, P_{-i}), P) \geq s(f(P'_i, P_{-i}), (P'_i, P_{-i})) \tag{42}$$

Also,

$$t_{k^\star}(P'_i) R'_i f(P). \tag{43}$$

Otherwise, $f(P) P'_i t_{k^\star}(P'_i)$ implies $s(f(P), P) = s(f(P), (P'_i, P_{-i}))$. By (42), $s(f(P'_i, P_{-i}, P)) = s(f(P), P)$ and $s(f(P), (P'_i, P_{-i})) = s(f(P'_i, P_{-i}), (P'_i, P_{-i}))$, contradicting the definition of f since $f(P'_i, P_{-i}) \neq f(P)$. So (43) holds.

Therefore, there exists $x \in X$ such that $t_{k^\star}(P_i) R_i x$ and $x P'_i t_{k^\star}(P'_i)$. As $k^\star n = m - 1$ is equivalent to $(n-1)k^\star = m - k^\star - 1$, we can consider $P^\star_{-i} \in \mathcal{P}^{n-1}$ such that the two following requirements hold: (i) $P^\star_j : x, f(P), \ldots$ for each $j \in N \setminus \{i\}$, and (ii) for each $y \in X \setminus \{f(P)\}$ such that $y P_i t_{k^\star}(P_i)$ there exist $j \in N \setminus \{i\}$ such that $t_{k^\star}(P^\star_j) R^\star_j y$. Therefore, since now $f(P)$ is the only alternative with score $n \cdot s_m$, $\mathcal{S}(P_i, P^\star_{-i}) = \{f(P)\}$ and

$$f(P_i, P^\star_{-i}) = f(P).$$

As $s(x, (P'_i, P^\star_{-i})) = n \cdot s_m$ and, by (43) and the definition of $P^\star_{-i}$, $s(r, (P'_i, P^\star_{-i})) < n \cdot s_m$ for each r such that $r P_i t_{k^\star}(P_i)$, it follows that $t_{k^\star}(P_i) R_i f(P'_i, P^\star_{-i})$ and we have

$$f(P_i, P^\star_{-i}) P_i f(P'_i, P^\star_{-i}). \tag{44}$$

By (41) and (44), f is regret-free truth-telling.

To see part (ii), let $f : \mathcal{P}^n \to X$ be a N-scoring rule. Then, there exists $\bar{j} \in N$ such that

$$f(\widetilde{P}) = \max_{\widetilde{P}_{\bar{j}}} \mathcal{S}(\widetilde{P}) \text{ for each } \widetilde{P} \in \mathcal{P}^n.$$

Let $P \in \mathcal{P}^n$. As we noted in the first paragraph of this proof, $s(f(P), P) = n \cdot s_m$and $f(P)P_j t_{k^\star}(P_j)$ for each $j \in N$.

Let $P_i' \in \mathcal{P}$ be such that

$$f(P_i', P_{-i})P_i f(P). \tag{45}$$

Since, by the definition of $k^\star$, $s(f(P_i', P_{-i}), P) = s(f(P), P) = n \cdot s_m$, it follows that $i \in N \setminus \{\bar{j}\}$.

Notice that, by the same arguments, (42) and (43) also hold here. Therefore, $t_{k^\star}(P_i)P_i' f(P)$ and there exists $x \in X$ such that $t_{k^\star}(P_i)P_i x$ and $xP_i' t_{k^\star}(P_i)$.

Now, let $P_{-i}^\star \in \mathcal{P}^{n-1}$ be such that $P_j^\star : x, f(P), \ldots$ for each $j \in N \setminus \{i\}$. As $s(f(P), (P_i, P_{-i}^\star)) = n \cdot s_m > s(x, (P_i, P_{-i}^\star))$ and $j \neq i$, $f(P_i, P_{-i}^\star) = f(P)$.As $s(x, (P_i', P_{-i}^\star)) = n \cdot s_m$ and $j \neq i$,$f(P_i', P_{-i}^\star) = x$. Then,

$$f(P_i, P_{-i}^\star)P_i f(P_i', P_{-i}^\star). \tag{46}$$

By (45) and (46), f is regret-free truth-telling. □

A.5 Proof of Theorem 5

Assume $n > 2$ and let $f : \mathcal{P}^n \to X$ be a scoring rule such that $k^\star n \geq m$ and $s_{k^\star-1} = s_{k^\star}$ (this implies $k^\star > 1$). If $k^\star = m - 1$, the result follows from Theorem 3, so assume that $k^\star < m - 1$. If f is and A-scoring rule, assume that a and b are the first two alternatives in the tie-breaking $\succ$ with $a \succ b$, whereas if f is a N-scoring rule, let agent 1 be the one who break ties. By the definition of $k^\star$, $s_{k^\star-1} = s_{k^\star} < s_{k^\star+1} = s_{m-1} = s_m$. Let $a, b \in X$. As $k^\star n \geq m$ and $k^\star > 1$, $k^\star(n-1) \geq m - k^\star$. Then, there exists $P \in \mathcal{P}^n$ such that:

(i) $b = t_{k^\star}(P_2)$ and $a = t_{k^\star-1}(P_2)$,

(ii) for each $j \in N \setminus \{2\}$, $t(P_j) = a$ and $t_{m-1}(P_j) = b$,

(iii) for each $x \in X$ such that $xP_2 b$, there exist $j \in N \setminus \{2\}$ such that $t_{k^\star}(P_j)R_j x$.

Since $s(a, P) \geq s(x, P)$ for each $x \in X$ such that $xR_2 b$, $a \succ x$ and $aP_1 b$, it follows that $bP_2 f(P)$. Let $P_2' \in \mathcal{P}$ be such that $t_{k^\star+1}(P_2') = b = t_{k^\star}(P_2)$, $t_{k^\star}(P_2') = t_{k^\star+1}(P_2)$, and $t_k(P_2') = t_k(P_2)$ for each $k \neq k^\star, k^\star + 1$. Let $\widehat{P} = (P_2', P_{-2})$. Then, by the definition of $k^\star$,

$$s(b, \widehat{P}) > s(b, P) = s(a, P) = s(a, \widehat{P})$$

and $s(b, \widehat{P}) \geq s(x, \widehat{P})$ if $bP_2 x$. Therefore,

$$f(P_2', P_{-2})P_2 f(P). \tag{47}$$

Let $P_{-2}^\star \in \mathcal{P}^{n-1}$ be such that $f(P_2, P_{-2}^\star) = f(P)$. Since $s(f(P), (P_2, P_{-2}^\star)) = s(f(P), (P_2', P_{-2}^\star))$ and $s(x, (P_2, P_{-2}^\star)) \geq s(x, (P_2', P_{-2}^\star))$ for each $x \in X \setminus \{b\}$, it follows that $f(P_2', P_{-2}^\star) \in \{f(P), b\}$. Therefore,

$$f(P_2', P_{-2}^\star)R_2 f(P_2, P_{-2}^\star). \tag{48}$$

By (47) and (48), f is not regret-free truth-telling. □

A.6 Proof of Theorem 6

Let $f : \mathcal{P}^n \to X$ be a Condorcet consistent and monotone rule. There are two cases to consider:

1. **$n \neq 2, 4$**. Then, there are $t \geq 1$ and $s \geq 0$ such that $n = 3t + 2s$. Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	$\cdots$	P_t	P_{t+1}	$\cdots$	P_{2t+s}	P_{2t+s+1}	$\cdots$	P_{3t+2s}
a	$\cdots$	a	b	$\cdots$	b	c	$\cdots$	c
b	$\cdots$	b	c	$\cdots$	c	a	$\cdots$	a
c	$\cdots$	c	a	$\cdots$	a	b	$\cdots$	b
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
t agents			$t+s$ agents			$t+s$ agents		

Since $C_P(a,c) = t < \frac{3t+2s}{2}$, $C_P(c,b) = t+s < \frac{3t+2s}{2}$, and $C_P(b,a) = t+s < \frac{3t+2s}{2}$, it follows that there is no Condorcet winner according to P.

Let $x = f(P)$. Then, there exists $i^\star \in N$ such that $x = t_k(P_{i^\star})$ with $k \leq m-2$. Assume first that $i^\star$ is such that $t+1 \leq i^\star \leq 2t+s$. Let $N' = \{j \in N : t+1 \leq j \leq 2t+s\}$ and consider the subprofile $P'_{N'} \in \mathcal{P}^{t+s}$ where, for each $j \in N'$, $P'_j \in \mathcal{P}$ is such that $t(P'_j) = c$, $t_{m-1}(P'_j) = b$, $t_{m-2}(P'_j) = a$, and $t_k(P'_j) = t_k(P_j)$ for each $k \leq m-3$. Then, c is the Condorcet winner in $(P'_{N'}, P_{-N'})$. As $i^\star \in N'$, $x \neq c$. This implies the existence of $S \subset N'$ and $j^\star \in N' \setminus S$ such that

$$f(P'_S, P_{-S}) = x \tag{49}$$

and

$$f(P'_{S\cup\{j^\star\}}, P_{-S\cup\{j^\star\}}) \neq x. \tag{50}$$

Now, by monotonicity and (50), $f(P'_{S\cup\{j^\star\}}, P_{-S\cup\{j^\star\}})P_{j^\star}x$, implying

$$f(P'_{S\cup\{j^\star\}}, P_{-S\cup\{j^\star\}})P_{j^\star}f(P'_S, P_{-S}). \tag{51}$$

Now let, $P^\star_{-j^\star} \in \mathcal{P}^{n-1}$ be such that $f(P_{j^\star}, P^\star_{-j^\star}) = f(P'_S, P_{-S})$. By (49), $f(P_{j^\star}, P^\star_{-j^\star}) = x$. Then, by monotonicity, $f(P'_{j^\star}, P^\star_{-j^\star})R_{j^\star}x$. Hence

$$f(P'_{j^\star}, P^\star_{-j^\star})R_{j^\star}f(P_{j^\star}, P^\star_{-j^\star}). \tag{52}$$

By (51) and (52), f is not regret-free truth-telling. The cases where $i^\star$ is such that $1 \leq i^\star \leq t$ or $2t+s+1 \leq i^\star \leq 3t+2s$ are similar and therefore we omit them.

2. **$n = 4$ and $m > 3$**. Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	P_2	P_3	P_4
a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$

As $C_P(a,c) = C_P(a,b) = C_P(b,d) = 2$, there is no Condorcet winner according to P. Let $f(P) = x$. Assume that $f(P) \notin \{b,c,d\}$ (the other 3 cases in which $f(P) \notin \{w,u,h\}$ with $\{w,u,h\} \subset \{a,b,c,d\}$ follow a similar argument). Next, let $P'_2 \in \mathcal{P}$ be such that $t(P'_2) = d$, $t_{m-1}(P'_2) = b$, $t_{m-2}(P'_2) = c$, $t_{m-3}(P'_2) = a$, and $t_k(P'_2) = t_k(P_2)$ for each $k \leq m-4$. Similarly, let $P'_3 \in \mathcal{P}$ be such that $t(P'_3) = d$, $t_{m-1}(P'_3) = c$, $t_{m-2}(P'_3) = a$, $t_{m-3}(P'_3) = b$, and $t_k(P'_3) = t_k(P_3)$ for each $k \leq m-4$. Then, d is the Condorcet winner according to $(P'_{\{2,3\}}, P_{-\{2,3\}})$. There are two cases to consider:

2.1. $\boldsymbol{f(P'_2, P_{-2}) \neq x}$. Then, by monotonicity, $f(P'_2, P_{-2})P_2x$. Hence,

$$f(P'_2, P_{-2})P_2f(P). \tag{53}$$

Now, let $P^{\star}_{-2} \in \mathcal{P}^{n-1}$ be such that $f(P_2, P^{\star}_{-2}) = f(P)$. Then, by monotonicity,

$$f(P'_2, P^{\star}_{-2})R_2f(P_2, P^{\star}_{-2}). \tag{54}$$

By (53) and (54), f is not regret-free truth-telling.

2.2. $\boldsymbol{f(P'_2, P_{-2}) = x}$. Then, $f(P'_{\{2,3\}}, P_{-\{2,3\}}) = dP_3x = f(P'_2, P_{-2})$ and an analogous reasoning to the one presented in Case 2.1 for agent 2, now performed with agent 3, shows that f is not regret-free truth-telling.

□

A.7 Proof of Corollary 5

We first show that each of the rules is monotone.

Lemma 1. *Simpson, Copeland, Young, Dodgson, Fishburn and Black rules (both anonymous and neutral) satisfy monotonicity.*

Proof. Let $x \in X$, $P \in \mathcal{P}^n$ and $P'_i \in \mathcal{P}$ be such that P'_i is a monotonic transformation of P_i with respect to x. Let $z \in X$ be such that xP_iz and let $y \in \{x,z\}$. Then, $C_P(y,a) = C_{(P'_i,P_{-i})}(y,a)$ for each $a \in X$. Therefore, (both anonymous and neutral) Simpson, Copeland and Fishburn rules are monotonic. To see that Young and Dodgson rules are monotonic, simply note that yP_ia if and only if yP'_ia for each $a \in X \setminus \{y\}$. Finally, to see that Black rule is monotonic, note that (i) y is a Cordorcet winner in P if and only if y is a Condorcet winner in (P'_i, P_{-i}), and (ii) the Borda score for y is the same in profiles P and (P'_i, P_{-i}). □

Proof of Corollary 5. Assume first that $N = \{1,2,3,4\}$ and $X = \{a,b,c\}$. In all of the cases that we consider in what follows, w.l.o.g., we assume that the tie-breaking is given by $a \succ b \succ c$ in the anonymous case, or by agent 1 in the neutral case.

Let $f : \mathcal{P}^4 \to \{a,b,c\}$ be a Simpson (Young, Dodgson, Fishburn) rule. Le $P \in \mathcal{P}^4$ be given by the following table:

P_1	P_2	P_3	P_4
b	c	c	a
a	b	b	c
c	a	a	b

Then, c is the only Simpson (Young, Dodgson, Fishburn) winner at P and $f(P) = c$. Now, consider $P_1' \in \mathcal{P}$ such that $P_1' : a, b, c$. Then, a is a Simpson (Young, Dodgson, Fishburn) winner at (P_1', P_{-1}). Therefore, $f(P_1', P_{-1}) = aP_1c = f(P)$. Let $P_{-1}^\star \in \mathcal{P}^{n-1}$ be such that $f(P_1, P_{-1}^\star) = f(P)$. Since $f(P) = c = t_1(P_1)$, $f(P_1', P_{-1}^\star) R_1 f(P_1, P_{-1}^\star)$. Hence, f is not regret-free truth-telling.

Next, let $f : \mathcal{P}^4 \to \{a,b,c\}$ be a Copeland (Black) rule. Let $P \in \mathcal{P}^4$ be given by

P_1	P_2	P_3	P_4
b	c	c	a
a	a	b	c
c	b	a	b

Then, c is the only Copeland (Black) winner at P and $f(P) = c$. Now, consider $P_1' \in \mathcal{P}$ such that $P_1' : a, b, c$. Then, a is a Copeland (Black) winner at (P_1', P_{-1}) and a similar reasoning to the one presented for Simpson' rule shows that f is not regret-free truth-telling.

Finally, assume $n \notin \{2,4\}$, or $n = 4$ and $m > 3$. By Lemma 1, Simpson, Copeland, Young, Dodgson, Fishburn and Black rules are monotonic. Since all of them are also Condorcet consistent, the result follows from Theorem 6. $\square$

A.8 Proof of Theorem 7

Let $f : \mathcal{P}^n \to X$ be a succesive elimination rule with associated order $a \succ b \succ c \succ \ldots$ and let $t \geq 1$ and $1 \geq s \geq 0$ be such that $n = 2t + s$. Next, let $P \in \mathcal{P}^n$ be given by the following table:[11]

P_1	P_2	P_3	$\cdots$	P_{t+2}	P_{t+3}	$\cdots$	P_{2t+s}
a	c	b	$\cdots$	b	a	$\cdots$	a
b	a	c	$\cdots$	c	b	$\cdots$	b
$\vdots$	b	a	$\cdots$	a	c	$\cdots$	c
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
			t agents			$t+s-2$ agents	

[11] Notice that, as $n \geq 3$, $t + s - 2 \geq 0$.

Since $C_P(a,b) = t+s \geq t = C_P(b,a)$, $C_P(a,c) = t+s-1 < t+1 = C_P(c,a)$, and $C_P(c,x) = n-1 > 1 = C_P(x,c)$ for each $x \in X \setminus \{a,b\}$, it follows that $f(P) = c$. Let $P_1' \in \mathcal{P}$ be such that $t(P_1') = b$, $t_{m-1}(P_1') = a$, and $t_1(P_1') = c$, and let $\widehat{P} = (P_1', P_{-1})$. Since $C_{\widehat{P}}(a,b) = t+s-1 < t+1 = C_{\widehat{P}}(b,a)$, $C_{\widehat{P}}(b,c) = n-1 > 1 = C_{\widehat{P}}(c,b)$, and $C_{\widehat{P}}(b,x) > C_{\widehat{P}}(x,b)$ for each $x \in X \setminus \{a,c\}$, it follows that $f(P_1', P_{-1}) = b$. Therefore,

$$f(P_1', P_{-1}) P_1 f(P). \tag{55}$$

Furthermore, as $f(P) = t_1(P_1)$,

$$f(P_1', P_{-1}^\star) R_1 f(P_1, P_{-1}^\star) \tag{56}$$

for each $P_{-1}^\star \in \mathcal{P}^{n-1}$ such that $f(P_1, P_{-1}^\star) = f(P)$. By (55) and (56), f is not regret-free truth-telling. □

A.9 Proof of Theorem 8

($\Longrightarrow$) Let $f : \mathcal{P}^2 \to \{a,b,c\}$ be a regret-free truth-telling and neutral rule.
Claim: f is efficient. Assume f is not efficient. W.l.o.g., there are two cases to consider:

1. **$P \in \mathcal{P}^2$ is such that $f(P) = c$, $P_i : a,b,c$, and P_j is such that $x = t(P_j) \neq c$.** By regret-free truth-telling, $f(P_i', P_j) = c$ for each $P_i' \in \mathcal{P}$. Let π be the permutation of X such that $\pi(c) = x$. By neutrality, $f(\pi P) = x$. Then, by regret-free truth-telling, $f(P_i', \pi P_j) = x$ for each $P_i' \in \mathcal{P}$. This implies that, as $f(\pi P) = x P_j c = f(\pi P_i, P_j)$ and $f(P_i', \pi P_j) = x$ for each $P_i' \in \mathcal{P}$, agent j manipulates f and does not regret it.

2. **$P \in \mathcal{P}^2$ is such that $f(P) = b$ and $P_i = P_j : a,b,c$.** Let π be the permutation of X such that $\pi(a) = b$. By neutrality, $f(\pi P) = a$. By the previous case, $f(\pi P_i, P_j) \neq c$. We claim that $f(\pi P_i, P_j) = b$. Assume $f(\pi P_i, P_j) = a$. Let $P_j^\star$ be such that $f(P_i, P_j^\star) = b$. If $f(\pi P_i, P_j^\star) = c$, then agent i manipulates f at $(\pi P_i, P_j^\star)$ via P_i and does not regret it. Therefore, $f(\pi P_i, P_j^\star) \neq c$. This implies that agent i manipulates f at P via πP_i and does not regret it. This proves the claim that $f(\pi P_i, P_j) = b$. By a similar reasoning to the one presented for agent i, we can see that agent j manipulates f at $(\pi P_i, P_j)$ via πP_j and does not regret it.

Since in both cases we reach a contradiction, f is efficient. This proves the claim.

Next, assume that f is not a dictatorship. We will prove that f is a N-maxmin rule. Let $\overline{P} \in \mathcal{P}^2$ be such that $\overline{P}_1 : a,b,c$ and $\overline{P}_2 : b,a,c$. By efficiency, $f(\overline{P}) \in \{a,b\}$. Assume, w.l.o.g., that $f(\overline{P}) = a$. We will prove that

$$f(P) = \max_{P_1} \mathcal{M}(P) \text{ for each } P \in \mathcal{P}^2.$$

Let $P \in \mathcal{P}^2$. There are three cases to consider:

1. **$t(P_1) = t(P_2)$.** By efficiency, $f(P) = t(P_1) = \max_{P_1} \mathcal{M}(P)$.

2. **$t(P_1) \neq t(P_2)$ and $t_1(P_1) = t_1(P_2)$.** As $f(\overline{P}) = a$, by neutrality, $f(P) = t(P_1) = \max_{P_1} \mathcal{M}(P)$.

3. **$t(P_1) \neq t(P_2)$ and $t_1(P_1) \neq t_1(P_2)$.** Then,

$$\mathcal{M}(P) = X \setminus \{t_1(P_1), t_1(P_2)\}. \tag{57}$$

If $f(P) = t_1(P_i) = x$ for some $i \in \{1, 2\}$, then $f(P) = t(P_j) = x$ with $j \neq i$ (because of efficiency). Then, by regret-free truth-telling,

$$f(P_j, P_i') = x \text{ for all } P_i'.$$

Then, again by regret-free truth-telling,

$$f(P_j', P_i') = x \text{ for all } P_i' \text{ and all } P_j' \text{ such that } t(P_j') = x.$$

Then, j is a dictator when he has top in x. Therefore, by neutrality, j is a dictator which is a contradiction. Thus,

$$f(P) \neq t_1(P_i) \text{ for all } i \in \{1, 2\}. \tag{58}$$

Therefore, by (57) and (58), $\mathcal{M}(P) = \{f(P)\}$ and $f(P) = \max_{P_1} \mathcal{M}(P)$.

($\Longleftarrow$) Let f be a N-maxmin rule. It is clear that f is neutral and, furthermore, by Theorem 1 (ii), f is regret-free truth-telling. If f is a dictatorship, it is trivial that it is neutral and regret-free truth-telling. □

A.10 Proof of Theorem 9

($\Longleftarrow$) Let $f : \mathcal{P}^2 \to \{a, b, c\}$ be a successive elimination rule or an A-maxmin* rule. It is clear that f satisfies efficiency and anonymity. We will prove that f is regret-free truth-telling. Assume there are $(P_1, P_2) \in \mathcal{P}^2$ and $P_1' \in \mathcal{P}$ such that

$$f(P_1', P_2) P_1 f(P_1, P_2). \tag{59}$$

We will prove that there exists $P_2^\star \in \mathcal{P}$ such that $f(P_1, P_2^\star) = f(P)$ and

$$f(P_1, P_2^\star) P_1 f(P_1', P_2^\star). \tag{60}$$

There are two cases to consider:

1. **f is a successive elimination rule with associated order $a \succ b \succ c$.** It is clear that

$$f(\widetilde{P}) \widetilde{R}_i a \text{ for each } \widetilde{P} \in \mathcal{P}^2 \text{ and each } i \in \{1, 2\}. \tag{61}$$

If $f(P_1, P_2) = a$, by (59) and efficiency, $a P_2 f(P_1', P_2)$, contradicting (61). Therefore, $f(P_1, P_2) P_1 a$ and, by (59), $t_1(P_1) = a$. There are two cases to consider:

1.1. $\boldsymbol{bP_1cP_1a}$. Then, by (59), $f(P_1, P_2) = c$ and, by definition of f, cP_2aP_2b. Therefore, there is no $P_1' \in \mathcal{P}$ such that $f(P_1', P_2) = b$, contradicting (59).

1.2. $\boldsymbol{cP_1bP_1a}$. Then, by (59), $f(P_1, P_2) = b$ and, by definition of f, $t(P_2) = b$. It follows from (59) that $f(P_1', P_2) = c$, implying that cP_2a and $cP_1'aP_1'b$. Now, let $P_2^\star \in \mathcal{P}$ be such that $bP_2^\star aP_2^\star c$. Then, $f(P_1, P_2^\star) = b$ and $f(P_1', P_2^\star) = a$. Since bP_1a, (60) holds and f is regret-free truth-telling.

2. **f is a A-maxmin* rule with associated binary relation $\succ^\star$.** By definition of f, it is clear that

$$f(\widetilde{P}) \neq t_1(\widetilde{P}_i) \text{ for each } \widetilde{P} \in \mathcal{P}^2 \text{ and each } i \in \{1,2\}. \tag{62}$$

W.l.o.g, let $P_1 : a, b, c$. By (59), $t(P_2) \neq a$ and $f(P) \neq a$. Then, by (62), $f(P) = b \in \{t(P_2), t_2(P_2)\}$. By (59) and (62), $f(P_1', P_2) = a \in \{t(P_2), t_2(P_2)\}$. Therefore, as $t(P_2) \neq a$, $P_2 : b, a, c$. Then, by definition of f, $b \succ^\star a$. Therefore, as $f(P_1', P_2) = a$, $t_1(P_1') = b$. Now let $P_2^\star : b, c, a$. Then, by (62), $f(P_1, P_2) = b = f(P_1, P_2^\star)$ and $f(P_1', P_2^\star) = c$. Since bP_1c, (60) holds and f is regret-free truth-telling.

($\Longrightarrow$) Assume that f is regret-free truth-telling, efficient, and anonymous. We will prove that f is a successive elimination rule or an A-maxmin* rule. There are two cases to consider:

1. **there exist $a \in X$ and $\overline{P} \in \mathcal{P}^2$ such that $f(\overline{P}) = t_1(\overline{P}_i) = a$ for some $i \in \{1,2\}$.** By efficiency, $f(\overline{P}) = t(\overline{P}_j) = a$ for $j = N \setminus \{i\}$. It follows, by regret-free truth-telling, that

$$f(P_i, \overline{P}_j) = a \text{ for each } P_i \in \mathcal{P}.$$

Then, again by regret-free truth-telling,

$$f(P) = a \text{ for each } P \in \mathcal{P}^2 \text{ such that } t(P_j) = a.$$

Therefore, by anonymity,

$$f(P) = a \text{ for all } P \text{ such that } a \in \{t(P_1), t(P_2)\}.$$

This implies, by regret-free truth-telling, that

$$f(P)R_ia \text{ for each } P \in \mathcal{P}^2 \text{ and each } i \in \{1,2\}. \tag{63}$$

Let $\widehat{P} \in \mathcal{P}^2$ be such that $\widehat{P}_1 : b, c, a$ and $\widehat{P}_2 : c, b, a$. By efficiency, $f(\widehat{P}) \in \{c, b\}$. W.l.o.g., assume that

$$f(\widehat{P}) = b. \tag{64}$$

Let $f^\succ$ be the successive elimination rule with associated order $a \succ b \succ c$ and let $P \in \mathcal{P}^2$. We will prove that $f = f^\succ$. There are two cases to consider:

1.1. **there exists $i \in \{1,2\}$ such that aP_ib or aP_ic.** Therefore, by (63), efficiency, and the definition of $f^\succ$, $f(P) = f^\succ(P)$.

1.2. ***bP_ia* and *cP_ia* for each *$i \in \{1,2\}$*.** If $t(P_1) = t(P_2)$, then by efficiency $f(P) = t(P_1) = f^{\succ}(P)$. Assume now that $t(P_1) \neq t(P_2)$. Then, by anonymity and (64), $f(P) = f(\widehat{P}) = b = f^{\succ}(P)$.

2. ***$f(P) \neq t_1(P_i)$* for each *$P \in \mathcal{P}^2$* and each *$i \in \{1,2\}$*.** First, let $\widehat{P} \in \mathcal{P}^2$ be such that $\widehat{P}_1 : b, c, a$ and $\widehat{P}_2 : c, b, a$. By efficiency, $f(\widehat{P}) \in \{b, c\}$. Assume, w.l.o.g., that $f(\widehat{P}) = b$. Second, let $\overline{P} \in \mathcal{P}^2$ be such that $\overline{P}_1 : b, a, c$ and $\overline{P}_2 : a, b, c$. By efficiency, $f(\overline{P}) \in \{b, a\}$. Assume, w.l.o.g., that $f(\widehat{P}) = a$. Third, let $\widetilde{P} \in \mathcal{P}^2$ be such that $\widetilde{P}_1 : c, a, b$ and $\widetilde{P}_2 : a, c, b$. By efficiency, $f(\widetilde{P}) \in \{c, a\}$. Assume, w.l.o.g., that $f(\widetilde{P}) = c$. We will prove that f is a A-maxmin rule* with associated binary relation $\succ^\star$ where $b \succ^\star c$, $a \succ^\star b$, and $c \succ^\star a$. This is, we need to show that

$$f(P) = \max_{\succ^\star} \mathcal{M}(P) \tag{65}$$

for each $P \in \mathcal{P}^2$. To do so, let $P \in \mathcal{P}^2$. If $P \in \{\widehat{P}, \overline{P}, \widetilde{P}\}$, it is clear that (65) holds. Assume $P \in \mathcal{P} \setminus \{\widehat{P}, \overline{P}, \widetilde{P}\}$. There are three cases to consider:

2.1. ***$t(P_1) = t(P_2)$*.** By efficiency, $f(P) = t(P_1) = \max_{\succ^\star} \mathcal{M}(P)$, so (65) holds.

2.2. ***$t_1(P_1) \neq t_1(P_2)$*.** As $|X| = 3$, there is $x \in X$ such that $\{x\} = X \setminus \{t_3(P_1), t_3(P_2)\}$. Therefore, as $f(P) \neq t_1(P_i)$ for each $i \in \{1,2\}$ (see hypothesis of Case 2), $f(P) = x$. Furthermore, as $t_3(P_1) \neq t_3(P_2)$, $\mathcal{M}(P) = \{x\}$ and then, (65) holds.

2.3. ***$t(P_1) \neq t(P_2)$* and *$t_1(P_1) = t_1(P_2)$*.** Then, $(P_1, P_2) = (P'_1, P'_2)$ with $P' \in \{\widehat{P}, \overline{P}, \widetilde{P}\}$. By anonymity and the fact that (65) holds for P',

$$f(P) = f(P') = \max_{\succ^\star} \mathcal{M}(P') = \max_{\succ^\star} \mathcal{M}(P).$$

Therefore, f is a successive elimination rule or an A-maxmin* rule, as stated. □